T-79.1001/2 **Syksy 2005**

Introduction to Theoretical Computer Science (T/Y)

Session 1

Answers to demonstration exercises

4 **Problem**: Prove that the following *de Morgan formulas* hold for all sets A, $B \subseteq U$:

$$\overline{A \cup B} = \overline{A} \cap \overline{B}, \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Answer: Two sets are equal when they have the same elements. Let us first examine the set $\overline{A \cup B}$. Let $a \in \overline{A \cup B}$. Then, $a \notin A \cup B$. By the definition of union this means that $a \notin A$ and $a \notin B$, so $a \in \overline{A}$ and $a \in \overline{B}$. This means that $a \in \overline{A} \cap \overline{B}$. Since every step in the proof preserves equivalence, the proof applies also for the other direction.

Next, consider $a \in \overline{A \cap B}$. Then, $a \notin A \cap B$ so $a \notin A$ or $a \notin B$. Now $a \in \overline{A}$ or $a \in \overline{B}$ so by definition of the union $a \in \overline{A} \cup \overline{B}$.

5. **Problem**: Define a relation $\sim$ on the set $\mathbb{N} \times \mathbb{N}$ by the rule:

$$(m, n) \sim (p, q) \quad \Leftrightarrow \quad m + n = p + q.$$

Prove that this is an equivalence relation, and describe intuitively ("geometrically") the equivalence classes it determines.

Solution: The relation $\backsim \subseteq (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$ is defined in the following way:

$$(m, n) \backsim (p, q) \quad \Leftrightarrow \quad m + n = p + q$$

In other words, two pairs are equivalent when their sums are the same.

A relation is an equivalence relation when it is symmetric, transitive and reflexive.

i) The relation $\backsim$ is symmetric, if $(m, n) \backsim (p, q)$ always when $(p, q) \backsim (m, n)$. Because

$$m + n = p + q \Leftrightarrow p + q = m + n,$$

$((p, q), (m, n))$ is always in the relation when $((m, n), (p, q))$ is. Thus the relation is symmetric.

ii) The relation $\backsim$ is reflexive, if for all $(m, n) \in \mathbb{N}$ holds that $(m, n) \backsim (m, n)$. Since

$$m + n = m + n,$$

the condition is fulfilled.

iii) The relation $\backsim$ is transitive, if always when $(m, n) \backsim (p, q)$ and $(p, q) \backsim (k, l)$, also $(m, n) \backsim (k, l)$.
Given

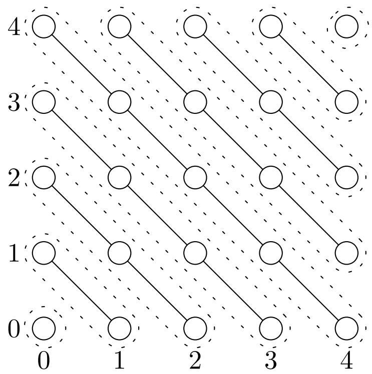
$$m + n = p + q \wedge p + q = k + l,$$

then

$$m + n = p + q = k + l \Rightarrow m + n = k + l,$$

and thus the relation is also transitive.

Because all three conditions hold, $\backsim$ is an equivalence relation. Below, the first elements of the relation as a graph.



From the figure it can be seen that the equivalence classes defined by the relation correspond with the lines that are parallel to the line $y = -x$.

6. **Problem**:Prove by induction that if X is a finite set of cardinality $n = |X|$, then its power set $\mathcal{P}(X)$ is of cardinality $|\mathcal{P}(X)| = 2^n$.

   Solution: Base case: $X = \emptyset$. Then $\mathcal{P}(\emptyset) = \{\emptyset\}$ and $|\mathcal{P}(\emptyset)| = 1 = 2^0$.

   Induction hypothesis: we assume there exists a $k \in \mathbb{N}$ such that formula holds for all $n \leq k$.

   Inductive step: let $|X| = k + 1$. Denote $X = Y \cup \{x\}$. By the induction hypothesis $|\mathcal{P}(Y)| = 2^k$. The set $\mathcal{P}(X)$ contains all elements of $\mathcal{P}(Y)$ and the union of the elements of $\mathcal{P}(Y)$ with $\{x\}$. Thus we get $|\mathcal{P}(X)| = 2 \cdot 2^k = 2^{k+1}$.