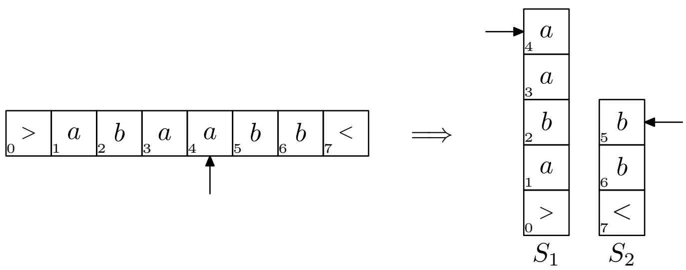
T-79.1001 **Syksy 2007**
Introduction to Theoretical Computer Science (T)
Session 9
Answers to demonstration exercises

4. **Problem**: Show that pushdown automata with two stacks (rather than just one as permitted by the standard definition) would be capable of recognizing exactly the same languages as Turing machines.

   Solution: We first show that a two-stack pushdown automaton can simulate a Turing machine. The only difficulty is to find a way to simulate the Turing machine tape using two stacks. This can be done using a construction that is similar to the one presented in the first problem: the first stack holds the part of tape that is left to the read/write head (in reversed order), and the second stack holds the symbols that are right to the head.



   The computation of the automaton can be divided into two parts:

   (a) Initialization, when the automaton copies the input to stack S_1 one symbol at a time, and then moves it, again one-by-one, to stack S_2. (With the exception of the first symbol).

   (b) Simulation, where the automaton decides its next transition by examining the top symbol of S_1. If the machine moves its head to left, the top element of S_1 is moved into S_2. If it moves to the other direction, the top element of S_2 is moved to S_1.

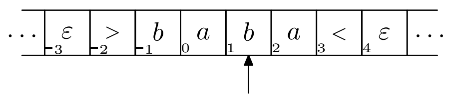
   A two-stack pushdown automaton that is formed using these principles simulates a given Turing machine. The formal details are presented in an appendix.

   Next we show that we can simulate a two-stack pushdown automaton using a Turing machine. This can be done trivially using a two tape nondeterministic Turing machine where both stacks are stored on their own tapes.

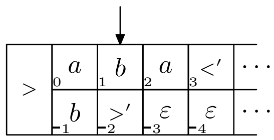
5. **Problem**: Extend the notion of a Turing machine by providing the possibility of a two-way infinite tape. Show that nevertheless such machines recognize exactly the same languages as the standard machines whose tape is only one-way infinite.

   Solution: A Turing machine with a two-way infinite tape works otherwise in a same way than a standard machine except that the position of the tape start symbol (>) is not fixed and it can move in a same way than the end symbol (<). The tape positions are indexed by the set $\mathbb{Z}$ of integers where 0 denotes the initial position of >.

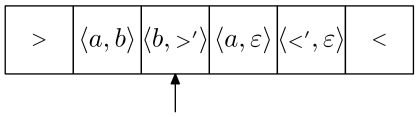
   We can simulate such a Turing machine by a two-track one-way Turing machine. Conceptually, we divide the tape into two parts: upper and lower. The upper part holds the two-way tape cells $i \geq 0$ and the lower part cells $i < 0$. For example, a two-way tape:



is expressed as a one-way tape:



In practice the construction of two tracks is done by replacing the alphabet Σ by a new alphabet $\Sigma' = (\Sigma \cup \{<', >'\}) \times (\Sigma \cup \{<', >'\})$. Each symbol of Σ' thus denotes two symbols of Σ. The symbols $\{<', >'\}$ are new symbols that denote the start and end symbols of the original tape. So, the above example is expressed as:



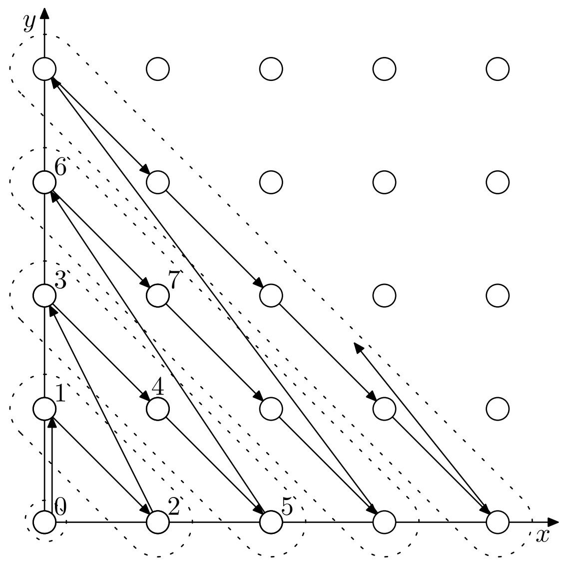
We still need a way to decide which tape-half is used. The easiest way to do this is to define a mirror image state q' for each state q. When the machine is in state q, it examines only the upper track when it decides what move to take next (tape head is on right side of the tape). Similarily, in state q' it examines only the lower symbol (tape head is on the left side). Since the lower tape is in a reversed order, all tape head moving instructions have to be also reversed.

The formal definition of this construction is presented in an appendix.

6. **Problem**: Prove that the Cartesian product $\mathbb{N} \times \mathbb{N}$ is countably infinite. (*Hint:* Think of the pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$ as embedded in the Euclidean (x, y) plane $\mathbb{R}^2$. Enumerate the pairs by diagonals parallel to the line $y = -x$.) Conclude from this result and the result of Problem 3 that also the set $\mathbb{Q}$ of rational numbers is countably infinite.

   Solution: A set S is countably infinite, if there exists a bijective mapping $f : \mathbb{N} \to S$. By intuition, all members of the set S can be assigned a unambiguous running number.

   The members $(x, y) \in \mathbb{N} \times \mathbb{N}$ of the set $\mathbb{N} \times \mathbb{N}$ can be assigned a number as shown in the following figure.



   The idea is to arrange all pairs of numbers on diagonals parallel to the line $y = -x$ and enumerate the lines by member one at a time, starting from the shortest one. Here the enumeration can not be done parallel to the x-axis; when doing this all indices would be used to enumerate only the y-axis and no pair $(x, y), y > 0$ would ever be reached.

The enumerating scheme abowe can be defined as follows:

$$f(x, y) = x + \sum_{k=1}^{x+y} k = x + \frac{(x + y)(x + y + 1)}{2}$$

For an example, $f(3, 1) = 13$, that is, the running number of pair $(3, 1)$ is 13. The function $f(x, y)$ is a bijection; for every running number there exists a unambiguous pair of numbers. Calculating a coordinate from a given index is relatively difficult, and is discussed in the appendix at the end of these solutions.

The set of positive rational numbers $\mathbb{Q}^+$ can be presented as a pair of numbers $\mathbb{N} \times \mathbb{N}$ by $(x, y) \equiv \frac{x}{y}, y \neq 0$. This is a proper subset of the countably infinite set $\mathbb{N} \times \mathbb{N}$. By Problem 3, $\mathbb{Q}^+$ is either finite or countably infinite. If $\mathbb{Q}^+$ was finite, there should exists some rational number $\frac{x}{y}, x \in \mathbb{N}, y \in \mathbb{N}, y \neq 0$, that would have the greatest running number $n < \infty$ (in the enumeration of $\mathbb{Q}$). This cannot be, because using the figure above one could always find a rational number that would have a running numberu $n' > n$. Hence, we have contradiction with the assumption that $\mathbb{Q}^+$ is finite. Therefore $\mathbb{Q}^+$ is countably infinite. By the same argument, the set $\mathbb{Q}^-$:

$$\mathbb{Q}^- = \{(-x, y) \mid (x, y) \in \mathbb{Q}^+\}$$

is countably infinite. Thus, the set $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^-$ is the union of two countably infinite sets, and it too is countably infinite.

Appendix: the formalisation of solution 5

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$ be a two-way tape Turing machine. Define a standard Turing machine M' as follows:

$$\begin{aligned}
M' =& (Q', \Sigma', \Gamma', \delta', q_0, q_{\text{acc}}, q_{\text{rej}}) \\
Q' =& Q \cup \{q' \mid q \in Q\} \\
\Sigma' =& (\Sigma \cup \{<', >'\}) \times (\Sigma \cup \{<', >'\}) \\
\Gamma' =& (\Gamma \cup \{<', >'\}) \times (\Gamma \cup \{<', >'\})
\end{aligned}$$

The transition function δ' is defined as follows:

$$\begin{aligned}
\delta' = & \{(q_1, \langle a, \gamma \rangle, q_2, \langle b, \gamma \rangle, \Delta) \mid (q_1, a, q_2, b, \Delta) \in \delta, \gamma \in \Gamma'\} \\
& \cup \{(q_1, \langle \sigma', \gamma \rangle, q_2, \langle b, \gamma \rangle, \Delta) \mid (q_1, \sigma, q_2, b, \Delta) \in \delta, \gamma \in \Gamma', \sigma \in \{<, >\}\} \\
& \cup \{(q_1', \langle \gamma, a \rangle, q_2', \langle \gamma, b \rangle, \overline{\Delta}) \mid (q_1, a, q_2, b, \Delta) \in \delta, \gamma \in \Gamma'\} \\
& \cup \{(q', \langle \gamma, a \rangle, q_{\text{end}}, \langle \gamma, b \rangle, \overline{\Delta}) \mid (q, a, q_{\text{end}}, b, \Delta) \in \delta, q_{\text{end}} \in \{q_{\text{acc}}, q_{\text{rej}}\}, \gamma \in \Gamma'\} \\
& \cup \{(q_1', \langle \gamma, \overline{\sigma}' \rangle, q_2', \langle \gamma, b \rangle, \overline{\Delta}) \mid (q_1, \sigma, q_2, b, \Delta) \in \delta, \gamma \in \Gamma', \sigma \in \{<, >\}\} \\
& \cup \{(q, >, q', >, R), (q', >, q, >, R) \mid q \in Q\},
\end{aligned}$$

where $\overline{L} = R$, $\overline{R} = L$, $\overline{<} = >$ and $\overline{>} = <$.