T-79.148 **Autumn 2003**
Introduction to Theoretical Computer Science
Tutorial 2
Solutions to the demonstration problems

4. **Problem**: Show that any alphabet Σ with at least two symbols is comparable to the binary alphabet $\Gamma = \{0, 1\}$, in the sense that strings over Σ can be easily encoded into strings over Γ and vice versa. How much can the length of a string change in your encoding? (I.e., if the length of a string $w \in \Sigma^*$ is $|w| = n$ symbols, what is the length of the corresponding string $w' \in \Gamma^*$?) Could you design a similar encoding if the target alphabet consisted of only *one* symbol, e.g. $\Gamma = \{1\}$?

   Solution: Let Σ be some alphabet with k symbols, $k > 1$. The strings of Σ can be coded as strings of $\Gamma = \{0, 1\}$ in the following manner.

   - Set the symbols of Σ to equal integers $\{1, \ldots, k\}$.
   - These numbers (the symbols of Σ) can be represented with binary numbers of length $\lceil \log_2 k \rceil$.
   - Every string in Σ^* can therefore be represented as a string of Γ by replacing the symbols of Σ with their binary encoding.

   The decoding from Γ^* to Σ^* can be done in a similar fashion by taking strings of length $\lceil \log_2 k \rceil$ from a string and interpreting them as symbols of Σ.

   If the length of a string $w \in \Sigma^*$ is $|w| = n$ symbols, the length of its counterpart $w' \in \Gamma^*$ is $|w'| = n \cdot \lceil \log_2 k \rceil$. This is because the number of symbols needed to encode any symbol in Σ is $\lceil \log_2 k \rceil$.

   For an example, consider the alphabet $\Sigma = \{a, b, c, d, e, f\}$ and the string $aacfd$. As $|\Sigma| = 6$, $\lceil \log_2 6 \rceil = \lceil 2.58 \rceil = 3$ bits are needed to represent the symbols of Σ. One possible encoding is

$$\begin{array}{ll} a \mapsto 001 & d \mapsto 100 \\ b \mapsto 010 & e \mapsto 101 \\ c \mapsto 011 & f \mapsto 110 \end{array}$$

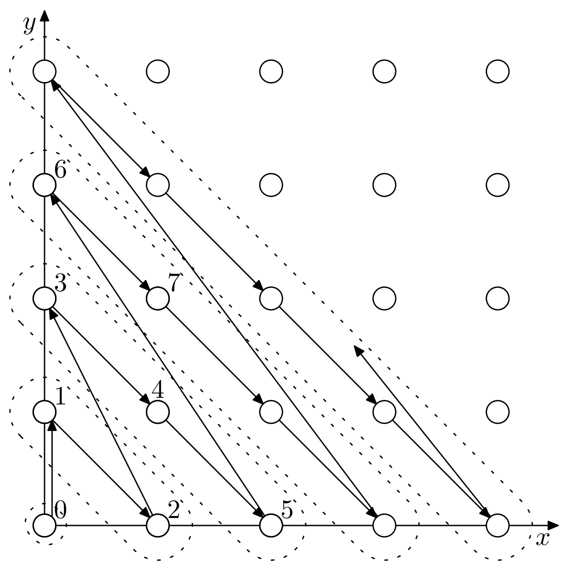
   With this encoding, the representation of $aacfd$ is 001001011110100.

   A similar coding scheme cannot be constructed if $\Gamma = \{1\}$. A unary presentation of the form $1 \mapsto 1$, $2 \mapsto 11$, $3 \mapsto 111$, ... can of course be defined, but the code obtained in this way can no longer be decoded unambiguously. For an example, the encodings of 1 1 1, 1 2, 2 1 and 3 are all the string 111.

5. **Problem**: Prove that the Cartesian product $\mathbb{N} \times \mathbb{N}$ is countably infinite. (*Hint:* Think of the pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$ as embedded in the Euclidean (x, y) plane $\mathbb{R}^2$. Enumerate the pairs by diagonals parallel to the line $y = -x$.) Conclude from this result and the result of Problem 3 that also the set $\mathbb{Q}$ of rational numbers is countably infinite.

   Solution: A set S is countably infinite, if there exists a bijective mapping $f : \mathbb{N} \to S$. By intuition, all members of the set S can be assigned a unambiguous running number.

   The members $(x, y) \in \mathbb{N} \times \mathbb{N}$ of the set $\mathbb{N} \times \mathbb{N}$ can be assigned a number as shown in the following figure.



The idea is to arrange all pairs of numbers on diagonals parallel to the line $y = -x$ and enumerate the lines by member one at a time, starting from the shortest one. Here the enumeration can not be done parallel to the x-axis; when doing this all indices would be used to enumerate only the y-axis and no pair $(x, y), y > 0$ would ever be reached.

The enumerating scheme abowe can be defined as follows:

$$f(x, y) = x + \sum_{k=1}^{x+y} k = x + \frac{(x + y)(x + y + 1)}{2}$$

For an example, $f(3, 1) = 13$, that is, the running number of pair $(3, 1)$ is 13. The function $f(x, y)$ is a bijection; for every running number there exists a unambiguous pair of numbers. Calculating a coordinate from a given index is relatively difficult, and is discussed in the appendix at the end of these solutions.

The set of positive rational numbers $\mathbb{Q}^+$ can be presented as a pair of numbers $\mathbb{N} \times \mathbb{N}$ by $(x, y) \equiv \frac{x}{y}, y \neq 0$. This is a proper subset of the countably infinite set $\mathbb{N} \times \mathbb{N}$. By Problem 3, $\mathbb{Q}^+$ is either finite or countably infinite. If $\mathbb{Q}^+$ was finite, there should exists some rational number $\frac{x}{y}, x \in \mathbb{N}, y \in \mathbb{N}, y \neq 0$, that would have the greatest running number $n < \infty$ (in the enumeration of $\mathbb{Q}$). This cannot be, because using the figure above one could always find a rational number that would have a running numberu $n' > n$. Hence, we have contradiction with the assumption that $\mathbb{Q}^+$ is finite. Therefore $\mathbb{Q}^+$ is countably infinite. By the same argument, the set $\mathbb{Q}^-$:

$$\mathbb{Q}^- = \{(-x, y) \mid (x, y) \in \mathbb{Q}^+\}$$

is countably infinite. Thus, the set $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^-$ is the union of two countably infinite sets, and it too is countably infinite.

6. **Problem**: Let S be an arbitrary nonempty set.

   (a) Give some injective (i.e. one-to-one) function $f : S \to \mathcal{P}(S)$.

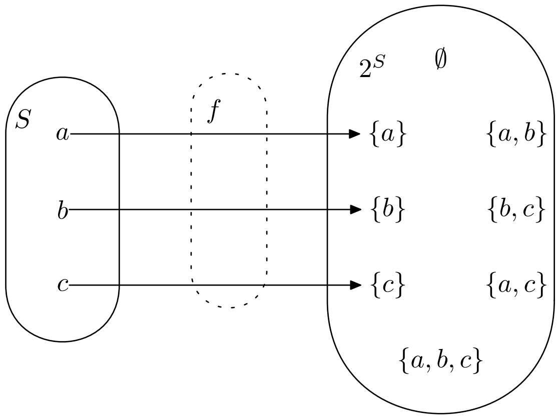
   (b) Prove that there cannot exist an injective function $g : \mathcal{P}(S) \to S$. (*Hint:* Assume that such a function g existed. Consider the set $R = \{s \in S \mid s \notin g^{-1}(s)\}$, and denote $r = g(R)$. Is it then the case that $r \in R$?)

Observe, as a consequence of item (b), that the power set $\mathcal{P}(S)$ of any countably infinite set S is uncountable.

Solution: Let $S \neq \emptyset$ be an arbitrary set.

(a) Define function $f : S \to \mathcal{P}(S)$ so that f is one-to-one (i.e. f is an injection). There exists $\{a\} \in \mathcal{P}(S)$ for all $a \in S$. Furthermore, if $a \neq b$, $\{a\} \neq \{b\}$ holds, so mapping $f : S \to \mathcal{P}(S)$, $f(a) = \{a\}$ is such an injection we were looking for.

Presented below is the mapping f for $S = \{a, b, c\}$:



(b) Assume that there exists an injection $g : \mathcal{P}(S) \to S$. First define a subset $S' \subseteq S$ such that:

$$S' = \{a \in S \mid \text{ there exists a set } A \subseteq S \text{ such that } g(A) = a\} \ .$$

We see that S' cannot be empty as $|\mathcal{P}(S)| > 0$ for all sets S.

Consider the set $R = \{s \ \in \ S' \ \mid s \notin g^{-1}(s)\}$, and denote $r = g(R)$. If $r \in R$ we have that $r \notin g^{-1}(r)$. However,

$$g^{-1}(r) = g^{-1}(g(R)) = R$$

so we have a contradiction. On the other hand, if $r \notin R$, it holds that $r \in g^{-1}(r) = g^{-1}(g(R)) = R$, another contradiction. Thus, it is not possible to define an injective function $g : \mathcal{P}(S) \to S$.

(The definition of S' above is necessary because otherwise the inverse g^{-1} would not necessarily exist in the definition of the set R.)

On the basis of (b) an injection $g : \ \mathcal{P}(S) \ \to \ S$ cannot be formed. Moreover, if S is countably infinite, there exists a bijection $f : \mathbb{N} \to S$. For $\mathcal{P}(S)$ to be countable, there should exists a bijection $f' : \ \mathbb{N} \ \to \ \mathcal{P}(S)$. Assume that such a bijection f' exists. Then mapping $g \circ f'^{-1} : \mathcal{P}(S) \to S$ a bijection (one-to-one and onto). This is contradictory with the fact that there exists no injection $\mathcal{P}(S) \ \to \ S$. Therefore $\mathcal{P}(S)$ is uncountable.

Appendix: Counting coordinate pairs from running numbers in problem 4.

Given a running number m one wishes to calculate such coordinates x and y that

$$x + \frac{(x+y)(x+y+1)}{2} = m \ . \tag{1}$$

Denote $z = x + y$. Then (1) equals to:

$$z - y + \frac{z(z+1)}{2} = m \ . \tag{2}$$

As $z - y \geq 0$,

$$\begin{aligned} \frac{z(z+1)}{2} &\leq m \quad \text{, it follows that} && (3) \\ z &\leq \frac{-1 \pm \sqrt{1+8m}}{2} && (4) \end{aligned}$$

As $z \in \mathbb{N}$ ja $m, x, y \geq 0$, it can be observed that:

$$z = \left\lfloor \frac{-1 + \sqrt{1+8m}}{2} \right\rfloor \tag{5}$$

Now both x and y can be calculated usind indexing function f:

$$\begin{aligned} x &= m - f(0, z) \\ y &= z - x \end{aligned} \tag{6}$$

Here $f(0, z)$ gives the running number of the first member of the diagonal (x, y). For an example, let us calculate the pair that corresponds to the running number $m = 13$:

$$\begin{aligned} z &= \left\lfloor \frac{-1 + \sqrt{105}}{2} \right\rfloor = \lfloor 4.62 \rfloor = 4 \\ x &= 13 - f(0, 4) = 13 - 10 = 3 \\ y &= 4 - 3 = 1 \end{aligned}$$

As a result the pair $(3, 1)$ is obtained.