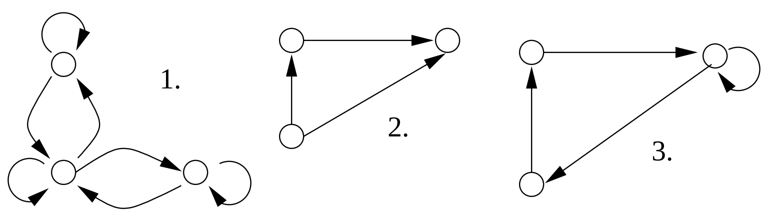
T-79.3001 Logic in computer science: foundations **Spring 2008**
Exercise 8 ([NS, 1997], Predicate Logic, Chapters 4 and 9)
April 2–4, 2008

Solutions to demonstration problems

Solution to Problem 4

The graphs given below illustrate different properties of relations. Here the nodes are the elements in a structure and there is an edge between two nodes $x \in A, y \in A$ if and only if $R(x,y)$ is true for x,y.



Reflexivity ($\forall x R(x,x)$) means that every node in the graph has an edge to itself and irreflexivity ($\forall x \neg R(x,x)$) means that no node has an edge to itself. First of the graphs is reflexive, the second irreflexive and the third is neither reflexive nor irreflexive.

Symmetry ($\forall x \forall y (R(x,y) \rightarrow R(y,x))$) means that whenever there is an edge from x to y, there is also an edge from y to x. Asymmetric ($\forall x \forall y (R(x,y) \rightarrow \neg R(y,x))$) graph has no edge from y to x if there is edge from x to y. The first graph is symmetric, the second asymmetric and the third is neither.

In a transitive graph ($\forall x \forall y \forall z (R(x,y) \wedge R(y,z) \rightarrow R(x,z))$) if there is a path from x to y along the edges, then there is an edge from x to y in the graph. The second graph is transitive.

In a serial graph ($\forall x \exists y R(x,y)$) there is at least one edge from each node x. The first and the third graph are serial.

Now define relations $T(x,y)$ (x knows y), $N(x,y)$ (x is married to y), $V(x,y)$ (y is a parent of x) ja $E(x,y)$ (y is an ancestor of x). There relations have the following properties.

Relation	refl.	irrefl.	symm.	asymm.	trans.	serial.
knows	*		*			*
married to		*	*			
parent		*		*		*
ancestor		*		*	*	*

Solution to Problem 5

a) Consider $\mathcal{S}$ with domain $U = \{1,2\}$ and $P^{\mathcal{S}} = \{\langle 1,1\rangle, \langle 2,2\rangle\}$. Now it holds $\mathcal{S} \models \forall x \exists y P(x,y)$ and $\mathcal{S} \not\models \exists y \forall x P(x,y)$ (there is no value for y such that for all x we would have $\langle x,y\rangle \in P^{\mathcal{S}}$). Thus the implication is false in $\mathcal{S}$.

b) Consider $\mathcal{S}$ with domain $U = \{1\}$ and $P^{\mathcal{S}} = \{1\}, Q^{\mathcal{A}} = \emptyset$. Now the left side of the implication is true and the right side false in $\mathcal{S}$, and $\mathcal{S}$ is a counterexample.

c) Consider $\mathcal{S}$ with domain $U = \{1\}$ ja $P^{\mathcal{S}} = \emptyset, R^{\mathcal{S}} = \{1\}$. Now $\forall x(P(x) \to R(x))$ is true in $\mathcal{S}$ since the left side of the implication is false in $\mathcal{S}$. Similarly $\mathcal{S} \models \forall x(P(x) \to \neg R(x))$.

Solution to Problem 6

- Remove connectives $\to$ and $\leftrightarrow$.
- Negations in, quantifiers out.
- Use distribution rules to obtain CNF / DNF.

a)

$$\begin{aligned}
&\forall y(\exists x P(x,y) \to \forall z Q(y,z)) \wedge \exists y(\forall x R(x,y) \vee \forall x Q(x,y)) \\
\equiv&\forall y(\neg\exists x P(x,y) \vee \forall z Q(y,z)) \wedge \exists y(\forall x R(x,y) \vee \forall x Q(x,y)) \\
\equiv&\forall y(\forall x \neg P(x,y) \vee \forall z Q(y,z)) \wedge \exists y(\forall x R(x,y) \vee \forall x Q(x,y)) \\
\equiv&\exists y_1(\forall y(\forall x \neg P(x,y) \vee \forall z Q(y,z)) \wedge (\forall x R(x,y_1) \vee \forall x Q(x,y_1))) \\
\equiv&\exists y_1 \forall y_2((\forall x \neg P(x,y_2) \vee \forall z Q(y_2,z)) \wedge (\forall x R(x,y_1) \vee \forall x Q(x,y_1))) \\
\equiv&\exists y_1 \forall y_2 \forall x_1 \forall x_2 \forall z \forall x_3((\neg P(x_1,y_2) \vee Q(y_2,z)) \wedge (R(x_2,y_1) \vee Q(x_3,y_1)))
\end{aligned}$$

This is the Prenex normal form and the part inside quantifiers is in CNF. Skolemization:

$$\forall y_2 \forall x_1 \forall x_2 \forall z \forall x_3((\neg P(x_1,y_2) \vee Q(y_2,z)) \wedge (R(x_2,c) \vee Q(x_3,c)))$$

c)

$$
\begin{aligned}
&\forall x \exists y Q(x,y) \vee (\exists x \forall y P(x,y) \wedge \neg \exists x \exists y P(x,y)) \\
\equiv &\forall x \exists y Q(x,y) \vee (\exists x \forall y P(x,y) \wedge \forall x \forall y \neg P(x,y)) \\
\equiv &\forall x \exists y Q(x,y) \vee \exists x_1 \forall y_1 \forall x_2 \forall y_2 (P(x_1,y_1) \wedge \neg P(x_2,y_2)) \\
\equiv &\exists x_1 \forall x_3 \exists y_3 \forall y_1 \forall x_2 \forall y_2 (Q(x_3,y_3) \vee (P(x_1,y_1) \wedge \neg P(x_2,y_2)))
\end{aligned}
$$

This is the Prenex normal form and we continue to get CNF.

$$\exists x_1 \forall x_3 \exists y_3 \forall y_1 \forall x_2 \forall y_2 ((Q(x_3,y_3) \vee P(x_1,y_1)) \wedge (Q(x_3,y_3) \vee \neg P(x_2,y_2)))$$

Skolemization:

$$\forall x_3 \forall y_1 \forall x_2 \forall y_2 ((Q(x_3, f(x_3)) \vee P(c,y_1)) \wedge (Q(x_3, f(x_3)) \vee \neg P(x_2,y_2)))$$

Solution to Problem 7

a)

$$
\begin{aligned}
&\forall x \phi(x) \rightarrow \psi \\
\equiv &\neg \forall x \phi(x) \vee \psi \\
\equiv &\exists x \neg \phi(x) \vee \psi \\
\equiv &\exists x_1 (\neg \phi(x_1) \vee \psi) \\
\equiv &\exists x_1 (\phi(x_1) \rightarrow \psi)
\end{aligned}
$$

b) Similarly, $\exists x \phi(x) \rightarrow \psi \equiv \forall x_1 (\phi(x_1) \rightarrow \psi)$.

c)

$$
\begin{aligned}
&\phi \rightarrow \forall x \psi(x) \\
\equiv &\neg \phi \vee \forall x \psi(x) \\
\equiv &\forall x_1 (\neg \phi \vee \psi(x_1)) \\
\equiv &\forall x_1 (\phi \rightarrow \psi(x_1))
\end{aligned}
$$

d) Similarly, $\phi \rightarrow \exists x \psi(x) \equiv \exists x_1 (\phi \rightarrow \psi(x_1))$.

Solution to Problem 8

a) Sentence $\neg\exists x((P(x) \to P(a)) \land (P(x) \to P(b)))$:
Eliminate implications: $\neg\exists x((\neg P(x) \lor P(a)) \land (\neg P(x) \lor P(b)))$.
Push $\neg$ inside $\exists x$:
$\forall x \neg((\neg P(x) \lor P(a)) \land (\neg P(x) \lor P(b)))$.
Push negations inside the formula:
$\forall x((P(x) \land \neg P(a)) \lor (P(x) \land \neg P(b)))$.
Bring $P(x)$ outside: $\forall x(P(x) \land (\neg P(a) \lor \neg P(b)))$.
Drop universal quantifiers: $P(x) \land (\neg P(a) \lor \neg P(b))$.
Clausal form: $\{\{P(x)\},\ \{\neg P(a), \neg P(b)\}\}$.

b) Sentence $\forall y \exists x P(x,y)$:
Skolemization: $\forall y P(f(y),y)$.
Drop universal quantifiers: $P(f(y),y)$.
Clausal form: $\{\{P(f(y),y)\}\}$.

c) Sentence $\neg\forall y \exists x G(x,y)$:
Push $\neg$ inside $\forall y$: $\exists y \neg\exists x G(x,y)$.
Push $\neg$ inside $\exists x$: $\exists y \forall x \neg G(x,y)$
Skolemization: $\forall x \neg G(x,c)$.
Drop universal quantifiers: $\neg G(x,c)$.
Clausal form: $\{\{\neg G(x,c)\}\}$.

d) Sentence $\exists x \forall y \exists z(P(x,z) \lor P(z,y) \to G(x,y))$:
Eliminate implication: $\exists x \forall y \exists z(\neg(P(x,z) \lor P(z,y)) \lor G(x,y))$.
Push negations inside:
$\exists x \forall y \exists z((\neg P(x,z) \land \neg P(z,y)) \lor G(x,y))$.
Push $G(x,y)$ inside the formula:
$\exists x \forall y \exists z((\neg P(x,z) \lor G(x,y)) \land (\neg P(z,y) \lor G(x,y)))$.
Skolemization: $\forall y \exists z((\neg P(c,z) \lor G(c,y)) \land (\neg P(z,y) \lor G(c,y)))$.
Skolemization: $\forall y((\neg P(c,f(y)) \lor G(c,y)) \land (\neg P(f(y),y) \lor G(c,y)))$.
Drop universal quantifiers:
$(\neg P(c,f(y)) \lor G(c,y)) \land (\neg P(f(y),y) \lor G(c,y))$.
Clausal form:
$\{\{\neg P(c,f(y)), G(c,y)\},\ \{\neg P(f(y),y), G(c,y)\}\}$.