12 Complexity of Search

- Combinatorial Phase Transitions
- Complexity of Local Search

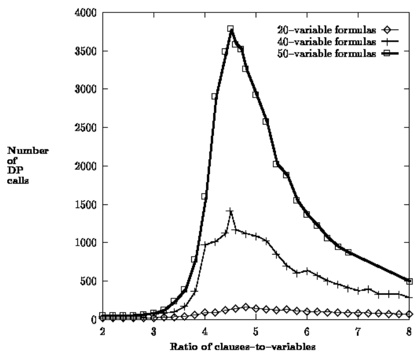
12.1 Combinatorial Phase Transitions

- "Where the Really Hard Problems Are" (Cheeseman et al. 1991)
- Many NP-complete problems can be solved in polynomial time "on average" or "with high probability" for reasonable-looking distributions of problem instances. E.g. Satisfiability in time $O(n^2)$ (Goldberg et al. 1982), Graph Colouring in time $O(n^2)$ (Turner 1988).
- Where, then, are the (presumably) exponentially hard instances of these problems located? Could one tell ahead of time whether a given instance is likely to be hard?
- Early studies: Yu & Anderson (1985), Hubermann & Hogg (1987), Cheeseman, Kanefsky & Taylor (1991), Mitchell, Selman & Levesque (1992), Kirkpatrick & Selman (1994), etc.

Hard instances for 3-SAT (1/4)

- Mitchell, Selman & Levesque, AAAI-92
- Experiments on the behaviour of the DPLL procedure on randomly generated 3-cnf Boolean formulas.
- Distribution of test formulas:
  - n = number of variables
  - $m = \alpha n$ randomly generated clauses of 3 literals, $2 \leq \alpha \leq 8$
- For sets of 500 formulas with $n = 20/40/50$ and various α, Mitchell et al. plotted the median number of recursive DPLL calls required for solution.

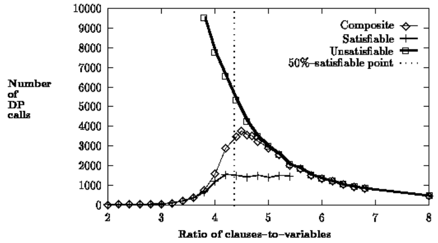
Hard instances for 3-SAT (2/4)



Results:

- A distinct peak in median running times at about clauses-to-variables ratio $\alpha \approx 4.5$.
- Peak gets more pronounced for increasing $n \Rightarrow$ well-defined "delta" distribution for infinite n?

Hard instances for 3-SAT (3/4)

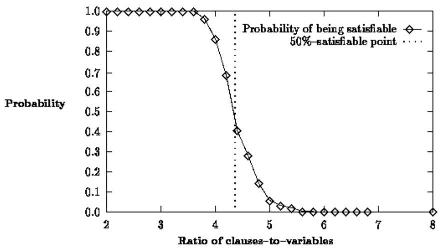


Further observations:

- The runtime peak seems to be located near the point where 50% of formulas are satisfiable.
- The peak seems to be caused by relatively short unsatisfiable formulas.

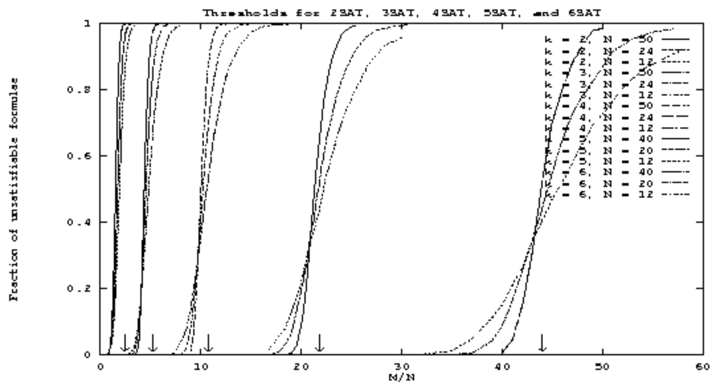
Question: Is the connection of the running time peak and the satisfiability threshold a characteristic of the DPLL algorithm, or a (more or less) algorithm independent "universal" feature?

The satisfiability transition (1/2)



Mitchell et al. (1992): The "50% satisfiable" point or "satisfiability threshold" for 3-SAT seems to be located at $\alpha \approx 4.25$ for large n.

The satisfiability transition (2/2)



Kirkpatrick & Selman (1994):

- Similar experiments as above for k-SAT, $k = 2, \ldots, 6$, 10000 formulas per data point.

- The "satisfiability threshold" α_c shifts quickly to larger values of α for increasing k.

Statistical mechanics of k-SAT (1/4)

Kirkpatrick & Selman, *Science* 1994

A "spin glass" model of a k-cnf formula:

- variables x_i $\sim$ spins with states ± 1
- clauses c $\sim$ k-wise interactions between spins
- truth assignment σ $\sim$ state of spin system
- Hamiltonian $H(\sigma)$ $\sim$ number of clauses unsatisfied by σ
- α_c $\sim$ critical "interaction density" point for "phase transition" from "satisfiable phase" to "unsatisfiable phase"

Statistical mechanics of k-SAT (2/4)

Estimates of α_c for various values of k via "annealing approximation", "replica theory", and observation:

k	α_{ann}	α_{rep}	α_{obs}
2	2.41	1.38	1.0
3	5.19	4.25	4.17 ± 0.03
4	10.74	9.58	9.75 ± 0.05
5	21.83	20.6	20.9 ± 0.1
6	44.01	42.8	43.2 ± 0.2

Statistical mechanics of k-SAT (3/4)

The "annealing approximation" means simply assuming that the different clauses are satisfied independently. This leads to the following estimate:

- Probability that given clause c is satisfied by random σ: $p_k = 1 - 2^{-k}$.
- Probability that random σ satisfies all $m = \alpha n$ clauses assuming independence: $p_k^{\alpha n}$.
- $E[\text{number of satisfying assignments}] = 2^n p_k^{\alpha n} \triangleq S_k^n(\alpha)$.
- For large n, $S_k^n(\alpha)$ falls rapidly from 2^n to 0 near a critical value $\alpha = \alpha_c$. Where is α_c?
- One approach: solve for $S_k^n(\alpha) = 1$.

$$S_k^n(\alpha) = 1 \Leftrightarrow 2p_k^{\alpha} = 1$$

$$\Leftrightarrow \alpha = -\frac{1}{\log_2 p_k} = -\frac{\ln 2}{\ln(1 - 2^{-k})} \approx \frac{\ln 2}{2^{-k}} = (\ln 2) \cdot 2^k$$

Statistical mechanics of k-SAT (4/4)

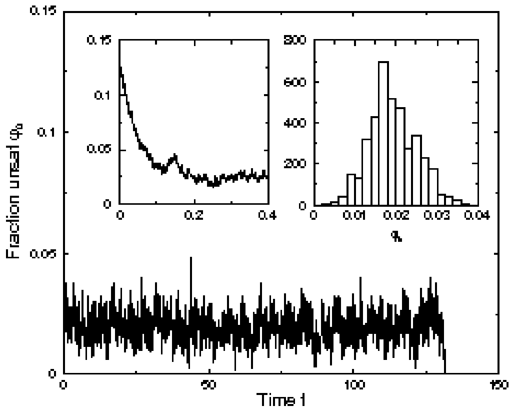
It is in fact known that:

- A sharp satisfiability threshold α_c exists for all $k \geq 2$ (Friedgut 1999).
- For $k = 2$, $\alpha_c = 1$ (Goerdt 1982, Chvátal & Reed 1982). Note that 2-SAT $\in$ P.
- For $k = 3$, $3.14 < \alpha_c < 4.51$ (lower bound due to Achlioptas 2000, upper bound to ???).
- Current best empirical estimate for $k = 3$: $\alpha_c \approx 4.267$ (Braunstein et al. 2002).
- For large k, $\alpha_c \sim (\ln 2) \cdot 2^k$ (Achlioptas & Moore 2002).

12.2 Complexity of Local Search

- Good experiences for 3-SAT in the satisfiable region $\alpha < \alpha_c$: e.g. GSAT (Selman et al. 1992), WalkSAT (Selman et al. 1996).
- *Focusing* the search on unsatisfied clauses seems to be an important technique: in the (unsystematic) experiments in Selman et al. (1996), WalkSAT (focused) outperforms NoisyGSAT (unfocused) by several orders of magnitude.

Dynamics of local search



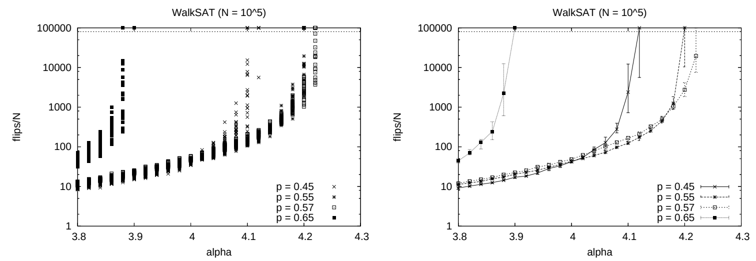
A WalkSAT run with $p = 1$ ("focused random walk") on a randomly generated 3-SAT instance, $\alpha = 3$, $n = 500$: evolution in the fraction of unsatisfied clauses (Semerjian & Monasson 2003).

Some recent results and conjectures

- Barthel, Hartmann & Weigt (2003), Semerjian & Monasson (2003): WalkSAT with $p = 1$ has a "dynamical phase transition" at $\alpha_{\text{dyn}} \approx 2.7 - 2.8$. When $\alpha < \alpha_{\text{dyn}}$, satisfying assignments are found in linear time per variable (i.e. in a total of cn "flips"), when $\alpha > \alpha_{\text{dyn}}$ exponential time is required.
- Explanation: for $\alpha > \alpha_{\text{dyn}}$ the search equilibrates at a nonzero energy level, and can only escape to a ground state through a large enough random fluctuation.
- Conjecture: all local search algorithms will have difficulties beyond the so called "clustering transition" at $\alpha \approx 3.92 - 3.93$ (Mézard, Monasson, Weigt et al.)

Some WalkSAT experiments

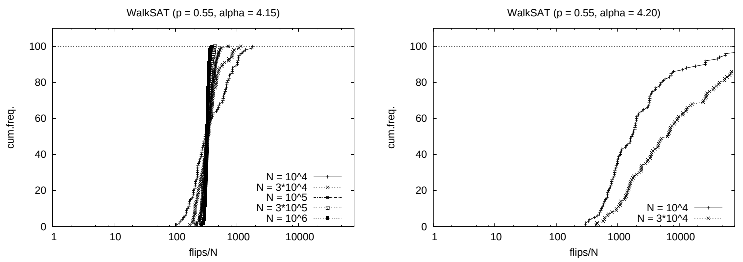
For $p > 1$, the α_{dyn} barrier for linear solution times can be broken (Aurell & Kirkpatrick 2004; Seitz, Alava & Orponen 2005).



Normalised (flips/n) solution times for finding satisfying assignments using WalkSAT, $\alpha = 3.8 \ldots 4.3$.
Left: complete data; right: medians and quartiles.

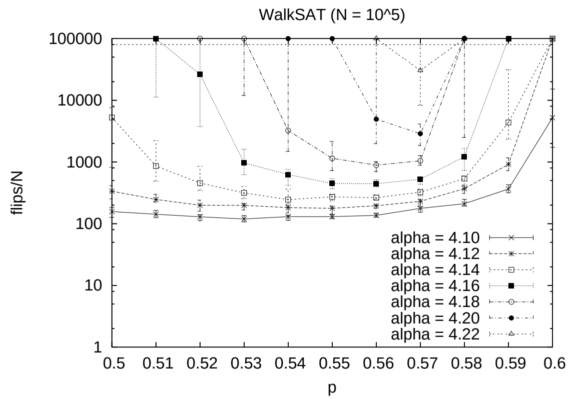
Data suggest linear solution times for $\alpha \gg \alpha_{\text{dyn}} \approx 2.7$.

WalkSAT linear scaling



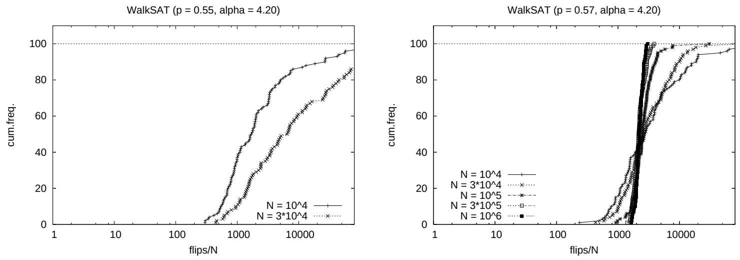
Cumulative solution time distributions for WalkSAT with $p = 0.55$.

WalkSAT optimal noise level?



Normalised solution times for WalkSAT with $p = 0.50 \ldots 0.60$, $\alpha = 4.10 \ldots 4.22$.

WalkSAT sensitivity to noise

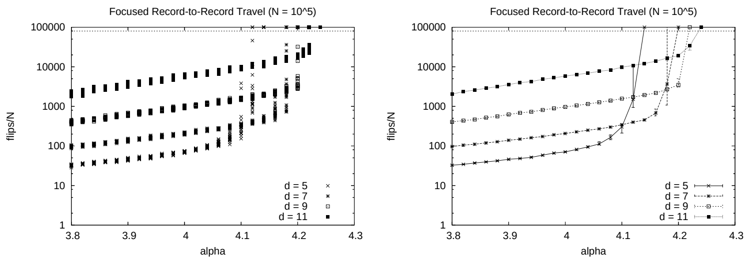


Cumulative solution time distributions for WalkSAT at $\alpha = 4.20$ with $p = 0.55$ and $p = 0.57$.

RRT applied to random 3-SAT

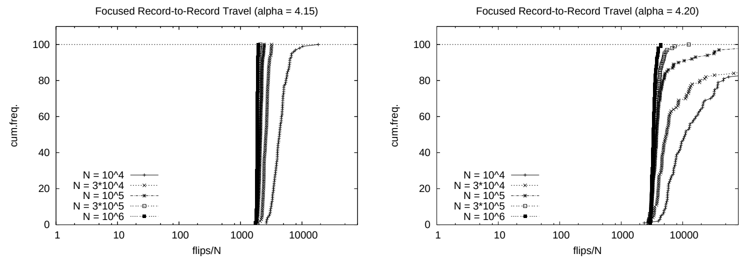
- Similar results as for WalkSAT are obtained with the Record-to-Record Travel algorithm.
- In applying RRT to SAT, $E(s) =$ number of clauses unsatisfied by truth assignment s. Single-variable flip neighbourhoods.
- *Focusing:* flipped variables chosen from unsatisfied clauses. (Precisely: one unsatisfied clause is chosen at random, and from there a variable at random.) $\Rightarrow$ FRRT = focused RRT.

FRRT experiments (3-SAT)



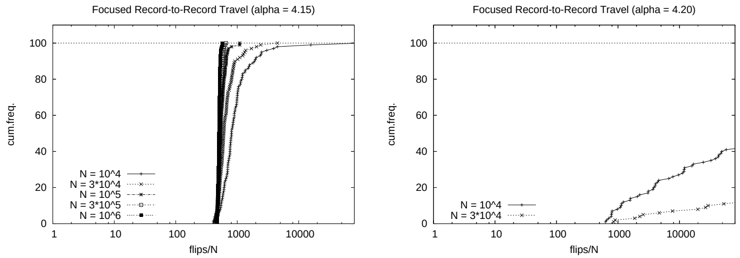
Normalised solution times for FRRT, $\alpha = 3.8 \ldots 4.3$.
Left: complete data; right: medians and quartiles.

FRRT linear scaling (1/2)



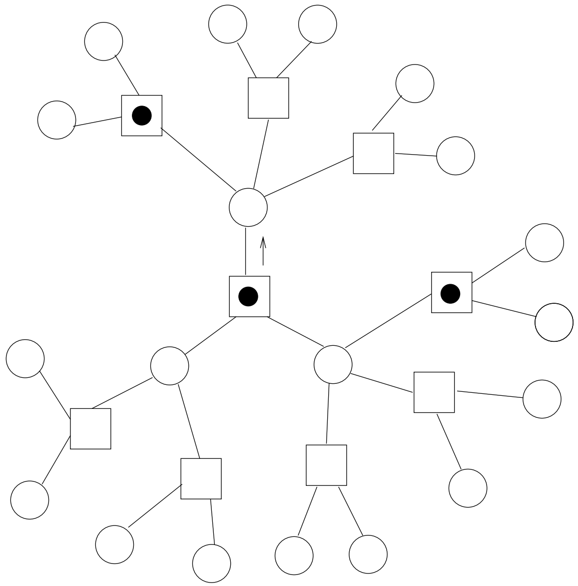
Cumulative solution time distributions for FRRT with $d = 9$.

FRRT linear scaling (2/2)

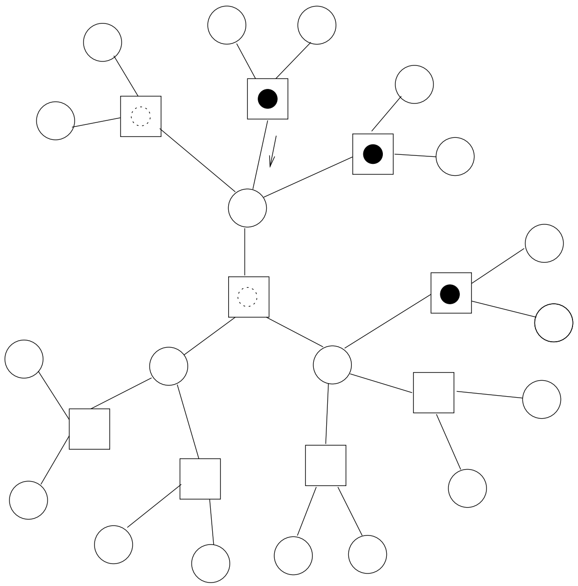


Cumulative solution time distributions for FRRT with $d = 7$.

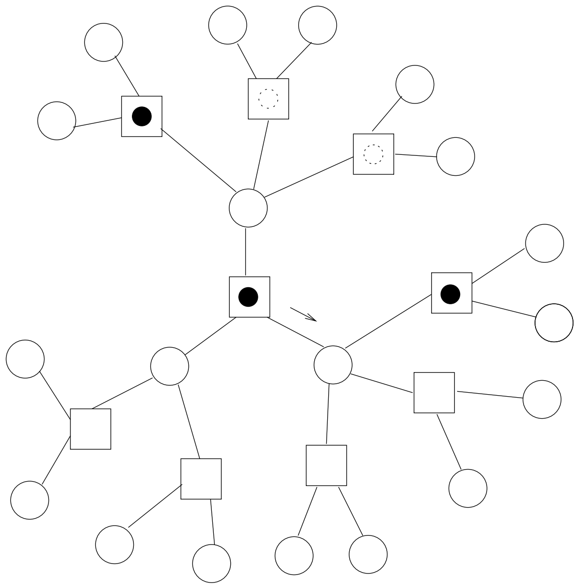
Focused search as a contact process



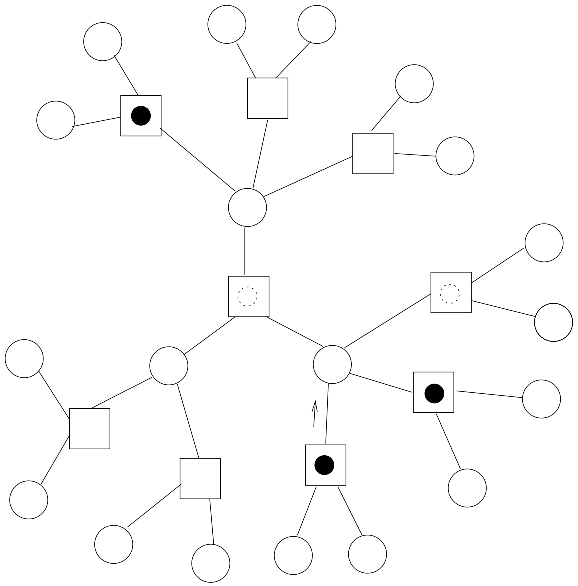
Focused search as a contact process



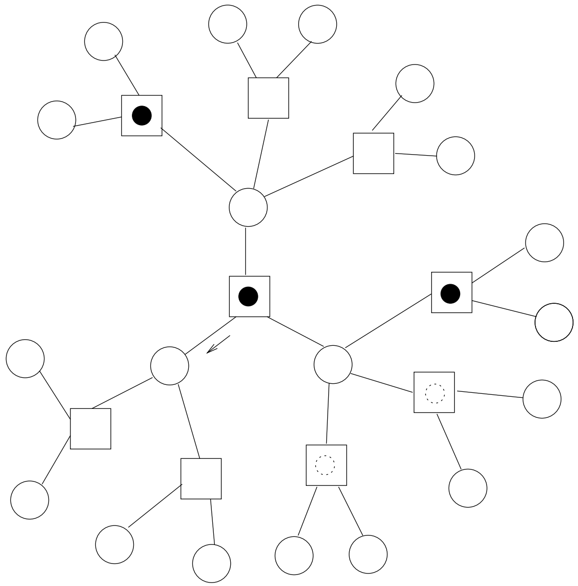
Focused search as a contact process



Focused search as a contact process



Focused search as a contact process



Focused search as a contact process