T-79.5101 **Spring 2008**
Advanced Course in Computational Logic
Exercise Session 1
Solutions

1. One can use analytic tableaux for finding a satisfying truth assignment for any satisfiable propositional logic formula by constructing a complete tableau starting from the given formula in the root. The set of atomic formulas in each of the open branches in such a complete tableau describes a satisfying truth assignment for the formula.

   a)

$$
\begin{array}{lccl}
1. & \multicolumn{2}{c}{\neg\big((P \to Q) \to (Q \to P)\big)} & \\
2. & \multicolumn{2}{c}{P \to Q} & (1) \\
3. & \multicolumn{2}{c}{\neg(Q \to P)} & (1) \\
4. & \multicolumn{2}{c}{Q} & (3) \\
5. & \multicolumn{2}{c}{\neg P} & (3) \\
6. & \neg P \quad (2) & \big|\; 7. \quad Q & (2)
\end{array}
$$

   In this complete tableau both of the open branches describe the same satisfying truth assignment, $M = \{Q\}$.

   b)

$$
\begin{array}{llll}
1. & \multicolumn{3}{c}{\big((P \vee \neg R) \leftrightarrow R\big) \wedge (P \to Q)} \\
2. & \multicolumn{2}{c}{(P \vee \neg R) \leftrightarrow R} & (1) \\
3. & \multicolumn{2}{c}{P \to Q} & (1) \\
4. & (P \vee \neg R) \wedge R \quad (2) & \big|\; 5. \quad \neg(P \vee \neg R) \wedge \neg R & (2) \\
11. & P \vee \neg R \quad (4) & \big|\; 6. \quad \neg(P \vee \neg R) & (5) \\
12. & R \quad (4) & \big|\; 7. \quad \neg R & (5) \\
13. & P \quad (11) \;\big|\; 14.\ \neg R\ (11) & \big|\; 8. \quad \neg P & (6) \\
15. & \neg P\ (3) \;\big|\; 16.\ Q\ (3) \;\big|\; \times & \big|\; 9. \quad \neg\neg R & (6) \\
 & \times & \big|\; 10. \quad R & (9) \\
 & & \quad\quad \times &
\end{array}
$$

   The only single branch in this complete tableau describes the satisfying truth assignment $M = \{P, Q, R\}$.

2. One can also use tableaux for investigating whether a given propositional logic formula ϕ is a logical consequence of a given set of propositional formulas Σ. Starting from the root which, in addition to all the formulas in Σ, contains $\neg\phi$, construct a complete tableau. If the tableau is closed, then ϕ is a logical consequence of Σ. On the other hand, any open branch in a complete tableau describes a counterexample for the logical consequence (a truth assignment that satisfies all the formulas in Σ but which does not satisfy ϕ).

$$
\begin{array}{lll}
1. & Q \to P & \\
2. & R \to (P \wedge Q) & \\
3. & P \to (Q \wedge R) & \\
4. & \neg\neg Q & \\
5. & Q & (4) \\
6. & \neg Q\ (1) \;\big|\; 7.\ P & (1) \\
 & \times \;\big|\; 8.\ \neg P\ (3) \;\big|\; 9.\ Q \wedge R & (3) \\
 & \quad \times \;\big|\; 10.\ Q & (9) \\
 & 11.\ R & (9) \\
 & 12.\ \neg R\ (2) \;\big|\; 13.\ P \wedge Q & (2) \\
 & \quad \times \;\big|\; 14.\ P & (13) \\
 & 15.\ Q & (13)
\end{array}
$$

   Since we have a complete tableau with an open branch, the formula $\neg Q$ is not a logical consequence of the given set of formulas Σ. From the open branch we obtain the countermodel $M = \{P, Q, R\}$.

3. A propositional logic formula is in conjunctive normal form (CNF) if it is of the form

$$(L_1^1 \vee \cdots \vee L_{n_1}^1) \wedge \cdots \wedge (L_1^m \vee \cdots \vee L_{n_m}^m),$$

   where each of the formulas L_i^j is a *literal* (an atomic formula or its negation).

   The disjunctive normal form (DNF) is

$$(L_1^1 \wedge \cdots \wedge L_{n_1}^1) \vee \cdots \vee (L_1^m \wedge \cdots \wedge L_{n_m}^m).$$

   One can apply tableaux to obtain the DNF of an arbitrary formula ϕ. Determine all satisfying truth assignments for ϕ by constructing a complete tableau with ϕ in the root, and then take the disjunction of these assignments.

1.		$(P \to Q) \to (P \lor Q)$		
2. $\neg(P \to Q)$	(1)	3. $P \lor Q$		(1)
4. P	(2)	6. P (3)	7. Q	(3)
5. $\neg Q$	(2)			

From this tableau we obtain the DNF $(P \land \neg Q) \lor P \lor Q$, which can be simplified further to $P \lor Q$.

For determining the CNF of ϕ, first form the DNF of $\neg\phi$:

1.	$\neg\big((P \to Q) \to (P \lor Q)\big)$		
2.	$P \to Q$		(1)
3.	$\neg(P \lor Q)$		(1)
4.	$\neg P$		(3)
5.	$\neg Q$		(3)
6.	$\neg P$ (2)	7. Q $\times$	(2)

Thus the DNF of the formula's negation

$$\neg\big((P \to Q) \to (P \lor Q)\big)$$

is

$$\neg P \land \neg Q.$$

The CNF of the original formula ϕ is then obtained by negating the DNF of $\neg\phi$ and applying De Morgan's rules. Now, the CNF of the formula

$$(P \to Q) \to (P \lor Q)$$

is

$$\neg(\neg P \land \neg Q) \equiv P \lor Q.$$

(In this example the CNF and DNF of the formula coincide. However, this does not hold in general.)

4. a)

1.	$\exists x_1 \exists x_2 P(x_1, x_2) \land \forall x_1 \forall x_2 \big(P(x_1, x_2) \to P(x_2, x_1)\big)$			
2.	$\exists x_1 \exists x_2 P(x_1, x_2)$			(1)
3.	$\forall x_1 \forall x_2 \big(P(x_1, x_2) \to P(x_2, x_1)\big)$			(1)
4.	$\exists x_2 P(c, x_2)$			$(2, x_1/c)$
5.	$P(c, d)$			$(4, x_2/d)$
6.	$\forall x_2 \big(P(c, x_2) \to P(x_2, c)\big)$			$(3, x_1/c)$
7.	$P(c, d) \to P(d, c)$			$(6, x_2/d)$
8. $\neg P(c, d)$ (7) $\times$	9. $P(d, c)$			(7)
	10. $\forall x_2 \big(P(d, x_2) \to P(x_2, d)\big)$			$(3, x_1/d)$
	11. $P(d, c) \to P(c, d)$			$(10, x_2/c)$
	12. $\neg P(d, c)$ (11) $\times$	13. $P(c, d)$		(11)
		14. $P(c, c) \to P(c, c)$		$(6, x_2/c)$
		15. $P(d, d) \to P(d, d)$		$(10, x_2/d)$
		16. $\neg P(c, c)$ (14)	17. $P(c, c)$	(14)
		18. $\neg P(d, d)$ (15) | 19. $P(d, d)$ (15)	20. $\neg P(d, d)$ (15) | 21. $P(d, d)$ (15)	

In this complete tableau there are four open branches. Based on each of these branches we can now construct a structure which gives a model for the given predicate logic formula in the root of the tableau. We'll now construct a structure $\mathcal{A}$ based on the leftmost open branch. Define the universe

$$A = \{1, 2\},$$

and

$$c^{\mathcal{A}} = 1, \qquad d^{\mathcal{A}} = 2 \quad \text{sekä} \quad P^{\mathcal{A}} = \{\langle 1, 2\rangle, \langle 2, 1\rangle\}.$$

Let's check that the given formula is true in the structure $\mathcal{A}$. Since e.g. $\langle 1, 2\rangle = \langle c^{\mathcal{A}}, d^{\mathcal{A}}\rangle \in P^{\mathcal{A}}$, we have $\mathcal{A} \models P(c, d)$, and hence

$$\mathcal{A} \models \exists x_1 \exists x_2 P(x_1, x_2)$$

holds. On the other hand

$$\mathcal{A} \models \forall x_1 \forall x_2 (P(x_1, x_2) \to P(x_2, x_1)),$$

holds as well since

$$\begin{aligned} &\mathcal{A} \models P(c, c) \to P(c, c), \\ &\mathcal{A} \models P(c, d) \to P(d, c), \\ &\mathcal{A} \models P(d, c) \to P(c, d) \text{ and} \\ &\mathcal{A} \models P(d, d) \to P(d, d), \end{aligned}$$

and $c^\mathcal{A} = 1, d^\mathcal{A} = 2, \langle c^\mathcal{A}, c^\mathcal{A}\rangle = \langle 1,1\rangle \notin P^\mathcal{A}$ (and hence $\mathcal{A} \not\models P(c,c)$), $\langle c^\mathcal{A}, d^\mathcal{A}\rangle = \langle 1,2\rangle \in P^\mathcal{A}$ (and hence $\mathcal{A} \models P(c,d)$), $\langle d^\mathcal{A}, c^\mathcal{A}\rangle = \langle 2,1\rangle \in P^\mathcal{A}$ (and hence $\mathcal{A} \models P(d,c)$), and $\langle d^\mathcal{A}, d^\mathcal{A}\rangle = \langle 2,2\rangle \notin P^\mathcal{A}$ ($\mathcal{A} \not\models P(d,d)$).

b) In this case there is no finite complete tableau; it turns out that the tableau rules force us to repeatedly introduce new constants, which then have to be repeatedly applied to the universally quantified formulas generated in the tableau.

(This exemplifies the *semi-decidability* of predicate logic: there is no systematic method using which, given an arbitrary predicate logic formula, one could either find a model or determine the formula as unsatisfiable in finitely many steps.)

However, the formula given in the exercise is true for example in the following structure $\mathcal{A}$:

Define the universe $A = \{1\}$. Additionally, we need a constant c and predicate P such that

$$c^\mathcal{A} = 1 \quad \text{and} \quad P^\mathcal{A} = \{\langle 1,1\rangle\}.$$

Let's check that

$$\mathcal{A} \models \forall x_1 \exists x_2 P(x_1,x_2) \land \forall x_1 \forall x_2 \forall x_3 (P(x_1,x_2) \land P(x_2,x_3) \to P(x_1,x_3))$$

holds. Since there is a single element in the universe ($c^\mathcal{A} = 1$) and $\langle 1,1\rangle \in P$ (and hence $\mathcal{A} \models P(c,c)$),

$$\mathcal{A} \models \forall x_1 \exists x_2 P(x_1,x_2)$$

holds. For the same reason

$$\mathcal{A} \models \forall x_1 \forall x_2 \forall x_3 (P(x_1,x_2) \land P(x_2,x_3) \to P(x_1,x_3)),$$

since

$$\mathcal{A} \models P(c,c) \land P(c,c) \to P(c,c).$$

5. a)

1.	$\neg\Big(\big(\forall x P(x) \land \forall x Q(x)\big) \to \forall x\big(P(x) \lor Q(x)\big)\Big)$	
2.	$\forall x P(x) \land \forall x Q(x)$	(1)
3.	$\neg\forall x\big(P(x) \lor Q(x)\big)$	(1)
4.	$\forall x P(x)$	(2)
5.	$\forall x Q(x)$	(2)
6.	$\neg\big(P(c) \lor Q(c)\big)$	$(3, x/c)$
7.	$\neg P(c)$	(6)
8.	$\neg Q(c)$	(6)
9.	$P(c)$	$(4, x/c)$
	$\times$	

b)

1.	$\neg\exists y\big(\exists x P(x) \to P(y)\big)$	
2.	$\neg\big(\exists x P(x) \to P(c)\big)$	$(1, y/c)$
3.	$\exists x P(x)$	(2)
4.	$\neg P(c)$	(2)
5.	$P(d)$	$(3, x/d)$
6.	$\neg\big(\exists x P(x) \to P(d)\big)$	$(1, y/d)$
7.	$\exists x P(x)$	(6)
8.	$\neg P(d)$	(6)
	$\times$	

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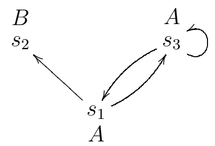
Exercise Session 2
Solutions

1. a) If φ is true, then the agent knows that φ.
   b) If the agent does not know that φ, then the agent knows that it does not know that φ.
   c) If the agent knows that ψ follows from φ, then it holds that if the agent knows that φ, then the agent knows that ψ.
   d) The agent knows that φ is true or the agent knows that φ is not true. In other words, the agent knows whether φ is true or not.

2. a) $\varphi \to LK\varphi$
   b) $L\varphi \wedge L\psi \to L(\varphi \wedge \psi)$
   c) $K\varphi \to L\varphi$
   d) $LL\varphi \to L\varphi$

3. Let $P =$ "it's raining".
   a) $K_aK_bP \wedge \neg K_bK_aK_bP$
   b) $K_a(\neg K_bP \wedge \neg K_b\neg P)$
   c) $K_b(K_aP \vee K_a\neg P)$
   d) $\neg K_aK_bK_aP \wedge \neg K_a\neg K_bK_aP$

4. We are given the model $\mathcal{M} = \langle S, R, v\rangle$:



   a) $\mathcal{M}, s_1 \Vdash \Box A$ does not hold because $\langle s_1, s_2\rangle \in R$ and $\mathcal{M}, s_2 \nVdash A$.
   b) $\mathcal{M}, s_1 \Vdash \Diamond B \to \Box\Diamond\top$ holds if and only if

$$\mathcal{M}, s_1 \nVdash \Diamond B \quad \text{or} \quad \mathcal{M}, s_1 \Vdash \Box\Diamond\top$$

holds. Since $\langle s_1, s_2\rangle \in R$ and $\mathcal{M}, s_2 \Vdash B$, $\mathcal{M}, s_1 \nVdash \Diamond B$ does not hold. On the other hand, $\mathcal{M}, s_1 \Vdash \Box\Diamond\top$ holds if and only if

$$\mathcal{M}, s_2 \Vdash \Diamond\top \quad \text{and} \quad \mathcal{M}, s_3 \Vdash \Diamond\top$$

holds. However, since there is no world $s \in S$ such that $\langle s_2, s\rangle \in R$, it follows that $\mathcal{M}, s_2 \nVdash \Diamond\top$, and hence $\mathcal{M}, s_1 \Vdash \Box\Diamond\top$ and $\mathcal{M}, s_1 \Vdash \Diamond B \to \Box\Diamond\top$ do not hold.

   c) $\mathcal{M}, s_3 \Vdash \Diamond\Diamond\Box\bot$ holds iff $\mathcal{M}, s_1 \Vdash \Diamond\Box\bot$ or $\mathcal{M}, s_3 \Vdash \Diamond\Box\bot$ holds. $\mathcal{M}, s_1 \Vdash \Diamond\Box\bot$ holds iff

$$\mathcal{M}, s_2 \Vdash \Box\bot \quad \text{or} \quad \mathcal{M}, s_3 \Vdash \Box\bot.$$

Since there is no world $s \in S$ such that $\langle s_2, s\rangle \in R$ it follows that $\mathcal{M}, s_2 \Vdash \Box\bot$ holds. Hence $\mathcal{M}, s_1 \Vdash \Diamond\Box\bot$ and, furthermore, $\mathcal{M}, s_3 \Vdash \Diamond\Diamond\Box\bot$ hold.

   d) $\mathcal{M}, s_1 \Vdash \Box(B \vee \Box\Diamond A)$ holds iff

$$\mathcal{M}, s_2 \Vdash B \vee \Box\Diamond A \quad \text{and} \quad \mathcal{M}, s_3 \Vdash B \vee \Box\Diamond A$$

hold. $\mathcal{M}, s_2 \Vdash B \vee \Box\Diamond A$ holds since $\mathcal{M}, s_2 \Vdash B$. $\mathcal{M}, s_3 \Vdash B \vee \Box\Diamond A$ holds iff

$$\mathcal{M}, s_3 \Vdash B \quad \text{or} \quad \mathcal{M}, s_3 \Vdash \Box\Diamond A.$$

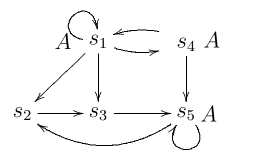
$\mathcal{M}, s_3 \Vdash B$ does not hold since $v(s_3, B) =$ false. Now $\mathcal{M}, s_3 \Vdash \Box\Diamond A$ holds iff $\mathcal{M}, s_1 \Vdash \Diamond A$ and $\mathcal{M}, s_3 \Vdash \Diamond A$ hold, which in turn is true since $\langle s_1, s_3\rangle \in R$, and $\langle s_3, s_3\rangle \in R$ and $v(s_3, A) =$ true. Hence, $\mathcal{M}, s_3 \Vdash \Box\Diamond A$ and $\mathcal{M}, s_3 \Vdash B \vee \Box\Diamond A$ hold. It follows that $\mathcal{M}, s_1 \Vdash \Box(B \vee \Box\Diamond A)$ holds.

   e) $\mathcal{M}, s_1 \Vdash \Diamond(\Box A \wedge \Box\neg A)$ holds iff

$$\mathcal{M}, s_2 \Vdash \Box A \wedge \Box\neg A \quad \text{or} \quad \mathcal{M}, s_3 \Vdash \Box A \wedge \Box\neg A.$$

Now, $\mathcal{M}, s_2 \Vdash \Box A \wedge \Box\neg A$ holds because $\mathcal{M}, s_2 \Vdash \Box A$ and $\mathcal{M}, s_2 \Vdash \Box\neg A$ since there is no world $s \in S$ for which $\langle s_2, s\rangle \in R$.

5. We are given the model $\mathcal{M} = \langle S, R, v\rangle$:



As in the previous exercise, we could determine the truth value of $\Box\Diamond\Box\Diamond A$ in each of the worlds by directly applying the definitions of $\Box$ and $\Diamond$. In other words, we could for example determine whether $\mathcal{M}, s_1 \Vdash \Box\Diamond\Box\Diamond A$, $\mathcal{M}, s_2 \Vdash \Box\Diamond\Box\Diamond A$, etc., until we find a world in which the given formula is true.

However, we can also use an alternative approach. Starting from the smallest subformula (which here is the atomic formula A), iteratively determine the truth values of the subformulas based on the truth values determined for the smaller subformulas in *each of the worlds* in the model. In the end, we have determined *all* worlds in the model where the formula $\Box\Diamond\Box\Diamond A$ itself is true. Any such world is an answer to the exercise.

Since $v(s_1, A) = v(s_4, A) = v(s_5, A) =$ true and $v(s, A) =$ false otherwise, we have that

$$\mathcal{M}, s_1 \Vdash A, \quad \mathcal{M}, s_4 \Vdash A \quad \text{and} \quad \mathcal{M}, s_5 \Vdash A$$

(and $\mathcal{M}, s \nVdash A$ otherwise). Since for example $\langle s_1, s_4 \rangle \in R$, $\langle s_3, s_5 \rangle \in R$, $\langle s_4, s_1 \rangle \in R$, and $\langle s_5, s_5 \rangle \in R$, by the semantics of $\Diamond$ it follows that

$$\mathcal{M}, s_1 \Vdash \Diamond A, \quad \mathcal{M}, s_3 \Vdash \Diamond A, \quad \mathcal{M}, s_4 \Vdash \Diamond A \ \text{ and } \ \mathcal{M}, s_5 \Vdash \Diamond A$$

hold. On the other hand, $\mathcal{M}, s_2 \Vdash \Diamond A$ does not hold since the only successor of the world s_2 in R is s_3 and $\mathcal{M}, s_3 \nVdash A$.

By the semactics of $\Box$ we have that

$$\mathcal{M}, s_2 \Vdash \Box\Diamond A, \quad \mathcal{M}, s_3 \Vdash \Box\Diamond A \quad \text{and} \quad \mathcal{M}, s_4 \Vdash \Box\Diamond A,$$

since for each world s' which is a successor of s_2, s_3, or s_4 we have that $\mathcal{M}, s' \Vdash \Diamond A$ holds. Additionally, these are the *only* worlds in which $\Box\Diamond A$ is true. (The formula $\Box\Diamond A$ is false in the worlds s_1 and s_5 since these worlds both have the successor s_2 and $\mathcal{M}, s_2 \nVdash \Diamond A$ holds.)

Now again by the semantics of $\Diamond$ we have

$$\mathcal{M}, s_1 \Vdash \Diamond\Box\Diamond A, \quad \mathcal{M}, s_2 \Vdash \Diamond\Box\Diamond A \quad \text{and} \quad \mathcal{M}, s_5 \Vdash \Diamond\Box\Diamond A$$

since each of the worlds s_1, s_2, and s_5 have a successor in which $\Box\Diamond A$ is true (since by the above we have $\mathcal{M}, s_2 \Vdash \Box\Diamond A$ and $\mathcal{M}, s_3 \Vdash \Box\Diamond A$, and e.g. $\langle s_1, s_2 \rangle \in R$, $\langle s_2, s_3 \rangle \in R$ and $\langle s_5, s_2 \rangle \in R$). Furthermore, $\Diamond\Box\Diamond A$ is false in the worlds s_3 (since the only successor of s_3 is s_5 but

$\mathcal{M}, s_5 \nVdash \Box\Diamond A$) and s_4 (since $\mathcal{M}, s_1 \nVdash \Box\Diamond A$ and $\mathcal{M}, s_5 \nVdash \Box\Diamond A$, and s_4 has no other successors).

Finally by the semantics of $\Box$ we have that

$$\mathcal{M}, s_3 \Vdash \Box\Diamond\Box\Diamond A, \quad \mathcal{M}, s_4 \Vdash \Box\Diamond\Box\Diamond A \quad \text{and} \quad \mathcal{M}, s_5 \Vdash \Box\Diamond\Box\Diamond A$$

hold since in each successor s' of the worlds s_3, s_4, and s_5 $\mathcal{M}, s' \Vdash \Diamond\Box\Diamond A$ holds. Now, s_3, s_4, and s_5 are the only worlds in which the formula $\Box\Diamond\Box\Diamond A$ is true.

Notice that at each stage it is important to determine *all* worlds in which the subformula at hand is true. Otherwise, we could end up in a situation in which the truth value of some other subformula could not be determined based directly on the already determined values. For example, if we would simple note that $\mathcal{M}, s_5 \Vdash A$ and $\mathcal{M}, s_3 \Vdash \Diamond A$ (since $\langle s_3, s_5 \rangle \in R$), we couldn't then determine the value of $\Box\Diamond A$ in world s_4 since it depends additionally on the values of $\Box\Diamond A$ in s_1 and s_5 (successors of s_4). Especially, it would then be a mistake to claim $\mathcal{M}, s_4 \nVdash \Box\Diamond A$.

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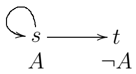
Exercise Session 3
Solutions

1. a) $\mathcal{M} = \langle S, R, v\rangle$, $S = \{s, t\}$, $R = \{\langle s, s\rangle, \langle s, t\rangle\}$, $v(s, A) = \text{true}$, $v(t, A) = \text{false}$.



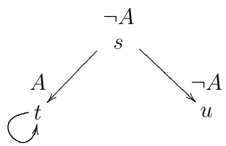
$\mathcal{M}, s \Vdash \Diamond A$ holds since $\langle s, s\rangle \in R$ and $\mathcal{M}, s \Vdash A$. $\mathcal{M}, s \Vdash \Box A$ does not hold since $\langle s, t\rangle \in R$ and $\mathcal{M}, t \nVdash A$. Hence $\mathcal{M}, s \nVdash \Diamond A \rightarrow \Box A$.

b) $\mathcal{M} = \langle S, R, v\rangle$, $S = \{s, t\}$, $R = \{\langle s, t\rangle\}$, $v(s, A) = v(t, A) = \text{false}$.

$$\neg A\ s \longrightarrow t\ \neg A$$

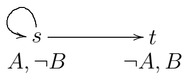
Since $\langle s, t\rangle \in R$ and $\mathcal{M}, t \nVdash A$, we have $\mathcal{M}, s \nVdash \Box A$. Hence, $\mathcal{M}, s \Vdash \neg\Box A$ holds. Since the world t has no successors, $\mathcal{M}, t \Vdash \Box A$ holds. Now $\mathcal{M}, t \nVdash \neg\Box A$ from which it follows that $\mathcal{M}, s \nVdash \Box\neg\Box A$ (since $\langle s, t\rangle \in R$). Hence $\mathcal{M}, s \nVdash \neg\Box A \rightarrow \Box\neg\Box A$.

c) $\mathcal{M} = \langle S, R, v\rangle$, $S = \{s, t, u\}$, $R = \{\langle s, t\rangle, \langle s, u\rangle, \langle t, t\rangle\}$, $v(s, A) = v(u, A) = \text{false}$ ja $v(t, A) = \text{true}$.



$\mathcal{M}, t \Vdash \Diamond A$ since $\langle t, t\rangle \in R$ and $\mathcal{M}, t \Vdash A$. Since t is itself its only successor, $\mathcal{M}, t \Vdash \Box A$ holds as well. Hence $\mathcal{M}, t \Vdash \Diamond A \wedge \Box A$ holds, from which it follows that $\mathcal{M}, s \Vdash \Diamond(\Diamond A \wedge \Box A)$ holds (since $\langle s, t\rangle \in R$). Since u has no successors, we have $\mathcal{M}, u \nVdash \Diamond A$. Since $\langle s, u\rangle \in R$, it follows that $\mathcal{M}, s \nVdash \Box\Diamond A$. Hence $\mathcal{M}, s \nVdash \Diamond(\Diamond A \wedge \Box A) \rightarrow \Box\Diamond A$.

d) $\mathcal{M} = \langle S, R, v\rangle$, $S = \{s, t\}$, $R = \{\langle s, s\rangle, \langle s, t\rangle\}$, $v(s, A) = v(t, B) = \text{true}$, $v(s, B) = v(t, A) = \text{false}$.



$\mathcal{M}, s \Vdash \Diamond A$ holds since $\langle s, s\rangle \in R$ and $\mathcal{M}, s \Vdash A$. $\mathcal{M}, s \Vdash \Diamond B$ holds since $\langle s, t\rangle \in R$ and $\mathcal{M}, t \Vdash B$. Hence $\mathcal{M}, s \Vdash \Diamond A \wedge \Diamond B$. $\mathcal{M}, s \Vdash \Diamond(A \wedge B)$ does not hold since s has no successor s' for which $\mathcal{M}, s' \Vdash A \wedge B$. Hence $\mathcal{M}, s \nVdash (\Diamond A \wedge \Diamond B) \rightarrow \Diamond(A \wedge B)$.

2. Let $\mathcal{M} = \langle S, R, v\rangle$.

($\Rightarrow$) Assume that $\Diamond\top$ is valid in $\mathcal{M}$. Take an arbitrary $s \in S$. By the assumption, $\mathcal{M}, s \Vdash \Diamond\top$ holds. Hence there is a $t \in S$ for which $\langle s, t\rangle \in R$ and $\mathcal{M}, t \Vdash \top$. Furthermore,

$$\text{if} \quad \mathcal{M}, s \Vdash \Box A, \quad \text{then also} \quad \mathcal{M}, t \Vdash A,$$

where t is a successor of s, and hence

$$\mathcal{M}, s \Vdash \Diamond A.$$

Thus, if $\mathcal{M}, s \Vdash \Box A$ then $\mathcal{M}, s \Vdash \Diamond A$. Hence $\mathcal{M}, s \Vdash \Box A \rightarrow \Diamond A$, and $\Box A \rightarrow \Diamond A$ is valid in $\mathcal{M}$.

($\Leftarrow$) Assume that $\Box A \rightarrow \Diamond A$ is valid in $\mathcal{M}$. Take an arbitrary $s \in S$. We claim that there is a $t \in S$ for which $\langle s, t\rangle \in R$. If this would not be the case, s would have no successors, in which case $\mathcal{M}, s \Vdash \Box A$ would hold. Since $\Box A \rightarrow \Diamond A$ is valid in $\mathcal{M}$ (by the assumption), now $\mathcal{M}, s \Vdash \Diamond A$ would also hold, which gives us a contradiction.

Hence there is a $t \in S$ for which $\langle s, t\rangle \in R$. Thus $\mathcal{M}, t \Vdash \top$ and therefore $\mathcal{M}, s \Vdash \Diamond\top$. It follow that $\Diamond\top$ is valid in $\mathcal{M}$.

3. We'll apply a known results for generated submodels[1]

> Given a model $\mathcal{M} = \langle S, R, v\rangle$, if $\mathcal{M}' = \langle S', R', v'\rangle$ is a submodel generated by $S_0 \subseteq S$, then for all formulas P and worlds $s \in S'$ it holds that
>
> $$\mathcal{M}, s \Vdash P \quad \text{iff} \quad \mathcal{M}', s \Vdash P.$$

[1] Let $\mathcal{M} = \langle S, R, v\rangle$ be a model. The *submodel $\mathcal{M}' = \langle S', R', v'\rangle$ generated by the set* $S_0 \subseteq S$ is a model which fulfills the following conditions.

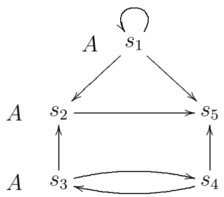
1. S' is the least subset of S for which the following hold:
   - $S_0 \subseteq S'$.
   - S' is closed under R: if $s \in S'$ and $t \in S$ for which $\langle s, t\rangle \in R$, then $t \in S'$.
2. $R' = (S' \times S') \cap R$.
3. $v'(s, P) = v(s, P)$ for all atomic formulas P and worlds $s \in S'$.

Since

$$\Box\big((\Box\Box A \to \Diamond\Box A)\wedge\Diamond\Box(\Box A \to \Diamond A)\big) \to \big(\Diamond(\Diamond A \to \Box A) \to ((\Diamond A\wedge\Box\Diamond A)\vee\Diamond\Box\Box\neg A)\big)$$

is true in the world s_4 in $\mathcal{M}$, the formula is valid in any generated submodel of $\mathcal{M}$ which contains the world s_4.

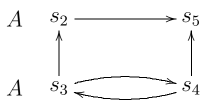
We are given the model $\mathcal{M}$:



Now, form the submodel $\mathcal{M}' = \langle S', R', v'\rangle$ generated by the set $S_0 = \{s_4\}$. Since $S_0 \subseteq S'$, we have that $s_4 \in S'$. Since $s_3 \in S$, $s_5 \in S$, and $\langle s_4, s_3\rangle \in R$, $\langle s_4, s_5\rangle \in R$, we have $s_3 \in S'$ and $s_5 \in S'$ (otherwise S' would not be closed under R). Furthermore, since $s_2 \in S$ and $\langle s_3, s_2\rangle \in R$, we have $s_2 \in S'$. Since the world s_1 is not reachable from any of the worlds s_2, s_3, s_4, s_5 under R, the set $\{s_2, s_3, s_4, s_5\}$ is closed under R. Clearly, this set is the smallest subset of S which is closed under R and contains the world s_3. Hence, to fulfill the requirements for a generated submodel, we define

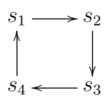
$$S' = \{s_2, s_3, s_4, s_5\},$$
$$R' = \{\langle s_2, s_5\rangle, \langle s_3, s_2\rangle, \langle s_3, s_4\rangle, \langle s_4, s_3\rangle, \langle s_4, s_5\rangle\}$$

and $v'(s_2, A) = v'(s_3, A) =$ true, $v'(s_4, A) = v'(s_5, A) =$ false. Now $\mathcal{M}'$ is



Since the formula given in the excercise is true in the world s_4 of the model $\mathcal{M}$, the formula is also true in the world s_4 in the model $\mathcal{M}'$ since $\mathcal{M}'$ is a generated submodel of $\mathcal{M}$. Furthermore, $\mathcal{M}'$ has four possible worlds as required.

4. $\mathcal{F} = \langle S, R\rangle$:



$\mathcal{F}' = \langle S', R'\rangle$, where $S' = \{r_1, r_2\}$ and $R' = \{\langle r_1, r_2\rangle,\ \langle r_2, r_1\rangle\}$:

Define the mapping $f : S \to S'$:

$$f(s_1) = f(s_3) = r_1$$
$$f(s_2) = f(s_4) = r_2$$

The mapping f is a p-morphism since

1. f is surjective (e.g., $r_1 = f(s_1)$ and $r_2 = f(s_2)$)
2. $\forall s, t \in S$: if sRt, then $f(s)R'f(t)$.
   (For example, corresponding to $\langle s_1, s_2\rangle \in R$ we have $\langle f(s_1), f(s_2)\rangle = \langle r_1, r_2\rangle$ which belongs to R'; one can make a similar check for all the pairs in R.)
3. $\forall s \in S\ \forall t \in S'$: if $f(s)R't$, then there is a $u \in S$ for which sRu and $f(u) = t$.
   (For example, $\langle r_2, r_1\rangle = \langle f(s_4), r_1\rangle \in R'$, and $s_1 \in S$, s_4Rs_1, and $f(s_1) = r_1$; the other cases are similar.)

By the proposition considering p-morphisms in the lecture notes it follows that

$$\text{if } \mathcal{F} \models P, \quad \text{then} \quad \mathcal{F}' \models P$$

for all formulas P.