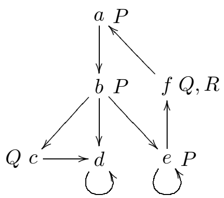
T-79.5101 **Spring 2008**
Advanced Course in Computational Logic
Exercise Session 11
Solutions

1. $\mathcal{M}$:



a) $\mathcal{M}, a \not\models \mathbf{A}(P\mathbf{U}Q)$, since (for example) the full path $(a, b, d, d, d, \ldots)$ starts from state a but does not go through any state $s \in S$ in which $\mathcal{M}, s \models Q$ holds.

b) A full path in a model is F-fair if and only if every $\varphi \in F$ is infinitely often true on the path.

Since $\{s \in S \mid \mathcal{M}, s \models R\} = \{f\}$, it follows that all F-fair paths in $\mathcal{M}$ must visit state f infinitely often. Since f is not reachably from neither of the states c and d, there is no F-fair path that visits these two states. Hence every F-fair path in $\mathcal{M}$ can be represented as

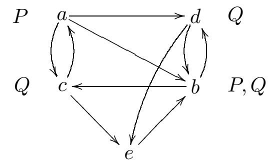
$$(a, b, \underbrace{e, \ldots, e}_{n_1 \text{ times}}, f, a, b, \underbrace{e, \ldots, e}_{n_2 \text{ times}}, f, a, b, \underbrace{e, \ldots, e}_{n_3 \text{ times}}, f, \ldots)$$

where $n_1, n_2, n_3, \ldots$ are (finite) positive integers. Especially, since n_1 is finite and, furthermore, $\mathcal{M}, a \models P$, $\mathcal{M}, b \models P$, $\mathcal{M}, e \models P$ and $\mathcal{M}, f \models Q$ hold, it follows that $\mathcal{M}, a \models_F \mathbf{A}(P\mathbf{U}Q)$ holds.

c) $\mathcal{M}, a \models \mathbf{EG}P$ holds since there is the full path $(a, b, e, e, e, \ldots)$ and for each state $s \in \{a, b, e\}$ we have $\mathcal{M}, s \models P$.

d) Notice that the path $(a, b, e, e, e, \ldots)$ is the only full path in which P holds and which starts from a. However, this path is not F-fair since it does not visit the state f. Hence $\mathcal{M}, a \not\models_F \mathbf{EG}P$.

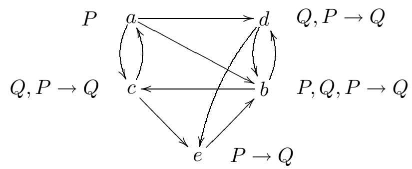
2. $\mathcal{M}$:



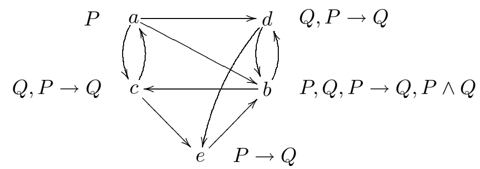
We sort the subformulas of $\mathbf{AXE}\big((P \to Q)\mathbf{U}(P \wedge Q)\big)$ so that the truth value of each subformula can be iteratively determined when the truth values of the preceding subformulas are known. One such order is

$$P, Q, P \to Q, P \wedge Q, \mathbf{E}\big((P \to Q)\mathbf{U}(P \wedge Q)\big), \mathbf{AXE}\big((P \to Q)\mathbf{U}(P \wedge Q)\big).$$

The truth values of the formulas P and Q are given directly by the valuation v. These in turn allow us to evaluate $P \to Q$ in each of the states of the model:



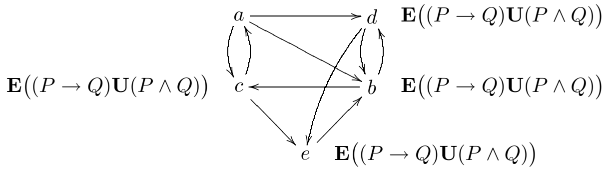
Then we consider the subformula $P \wedge Q$:



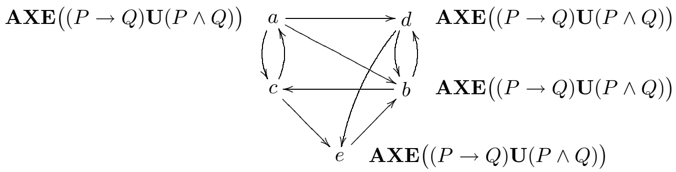
Next we evaluate $\mathbf{E}\big((P \to Q)\mathbf{U}(P \wedge Q)\big)$ using the algorithm CheckEU in the lecture notes. We start from the set of states in which $P \wedge Q$ is true ($\{b\}$) and mark $\mathbf{E}\big((P \to Q)\mathbf{U}(P \wedge Q)\big)$ as true in those states. Then we collect all states $s \in S$ such that sRb and $\mathcal{M}, s \models P \to Q$ (disregarding those states in which $\mathbf{E}\big((P \to Q)\mathbf{U}(P \wedge Q)\big)$ is already marked as true). We arrive at the set of states $\{d, e\}$, and hence mark $\mathbf{E}\big((P \to Q)\mathbf{U}(P \wedge Q)\big)$ as true in these states. Repeat this for the predecessors of d and e, and again for the predecessors of the states we arrive at, iteratively until we do not arrive at any new states. This procedure can be described as follows.

Round	Visited states	Considered states	New states
1	$\{b\}$	$\{b\}$	$\{d,e\}$
2	$\{b,d,e\}$	$\{d,e\}$	$\{c\}$
3	$\{b,c,d,e\}$	$\{c\}$	$\emptyset$

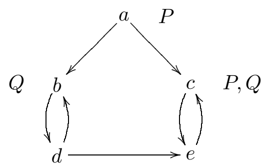
We now know that $\mathbf{E}\big((P \to Q)\mathbf{U}(P \wedge Q)\big)$ is true precisely in the set of states $\{b,c,d,e\}$.



Finally, we can evaluate the formula $\mathbf{AXE}\big((P \to Q)\mathbf{U}(P \wedge Q)\big)$: the formula is true in a state $s \in S$ if and only if $\mathbf{E}\big((P \to Q)\mathbf{U}(P \wedge Q)\big)$ is true in all successors of s. Hence we arrive at



3. $\mathcal{M}$:



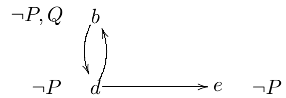
For the evaluation we employ the CheckEG and CheckEU algorithms. First, we have to express the formula using the operators $\mathbf{EU}$ and $\mathbf{EG}$:

$$
\begin{aligned}
&\mathbf{AG}\big(Q \to \mathbf{A}(\mathbf{EF}P\mathbf{U}\mathbf{AF}P)\big)\\
\equiv\ &\mathbf{AG}\big(Q \to \mathbf{A}(\mathbf{E}(\top\mathbf{U}P)\mathbf{U}\neg\mathbf{EG}\neg P)\big)\\
\equiv\ &\mathbf{AG}\big(Q \to \neg\mathbf{E}\big((\neg\neg\mathbf{EG}\neg P)\mathbf{U}(\neg\mathbf{E}(\top\mathbf{U}P) \wedge \neg\neg\mathbf{EG}\neg P)\big) \wedge \neg\mathbf{EG}\neg\neg\mathbf{EG}\neg P\big)\\
\equiv\ &\mathbf{AG}\big(Q \to \neg\mathbf{E}\big((\mathbf{EG}\neg P)\mathbf{U}(\neg\mathbf{E}(\top\mathbf{U}P) \wedge \mathbf{EG}\neg P)\big) \wedge \neg\mathbf{EGEG}\neg P\big)\\
\equiv\ &\neg\mathbf{EF}\neg\big(Q \to \neg\mathbf{E}\big((\mathbf{EG}\neg P)\mathbf{U}(\neg\mathbf{E}(\top\mathbf{U}P) \wedge \mathbf{EG}\neg P)\big) \wedge \neg\mathbf{EGEG}\neg P\big)\\
\equiv\ &\neg\mathbf{E}\big(\top\mathbf{U}\neg\big(Q \to \neg\mathbf{E}\big((\mathbf{EG}\neg P)\mathbf{U}(\neg\mathbf{E}(\top\mathbf{U}P) \wedge \mathbf{EG}\neg P)\big) \wedge \neg\mathbf{EGEG}\neg P\big)\big)
\end{aligned}
$$

Then, sort the subformulas into a suitable order.

$$
\begin{gathered}
P, Q, \neg P, \mathbf{EG}\neg P, \mathbf{EGEG}\neg P, \neg\mathbf{EGEG}\neg P,\\
\mathbf{E}(\top\mathbf{U}P), \neg\mathbf{E}(\top\mathbf{U}P), \neg\mathbf{E}(\top\mathbf{U}P) \wedge \mathbf{EG}\neg P,\\
\mathbf{E}\big((\mathbf{EG}\neg P)\mathbf{U}(\neg\mathbf{E}(\top\mathbf{U}P) \wedge \mathbf{EG}\neg P)\big), \neg\mathbf{E}\big((\mathbf{EG}\neg P)\mathbf{U}(\neg\mathbf{E}(\top\mathbf{U}P) \wedge \mathbf{EG}\neg P)\big),\\
\neg\mathbf{E}\big((\mathbf{EG}\neg P)\mathbf{U}(\neg\mathbf{E}(\top\mathbf{U}P) \wedge \mathbf{EG}\neg P)\big) \wedge \neg\mathbf{EGEG}\neg P,\\
Q \to \neg\mathbf{E}\big((\mathbf{EG}\neg P)\mathbf{U}(\neg\mathbf{E}(\top\mathbf{U}P) \wedge \mathbf{EG}\neg P)\big) \wedge \neg\mathbf{EGEG}\neg P,\\
\neg\big(Q \to \neg\mathbf{E}\big((\mathbf{EG}\neg P)\mathbf{U}(\neg\mathbf{E}(\top\mathbf{U}P) \wedge \mathbf{EG}\neg P)\big) \wedge \neg\mathbf{EGEG}\neg P\big),\\
\mathbf{E}\big(\top\mathbf{U}\neg\big(Q \to \neg\mathbf{E}\big((\mathbf{EG}\neg P)\mathbf{U}(\neg\mathbf{E}(\top\mathbf{U}P) \wedge \mathbf{EG}\neg P)\big) \wedge \neg\mathbf{EGEG}\neg P\big)\big),\\
\neg\mathbf{E}\big(\top\mathbf{U}\neg\big(Q \to \neg\mathbf{E}\big((\mathbf{EG}\neg P)\mathbf{U}(\neg\mathbf{E}(\top\mathbf{U}P) \wedge \mathbf{EG}\neg P)\big) \wedge \neg\mathbf{EGEG}\neg P\big)\big)
\end{gathered}
$$

Since P is true in the states a and c, $\neg P$ is true in the states $\{b,d,e\}$. Now we can apply the algorithm CheckEG to evaluate $\mathbf{EG}\neg P$. First, construct the restriction $\mathcal{M}'$ based on the states in which $\neg P$ is true.

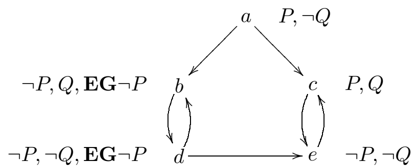


Then, find the non-trivial strongly connected components (SCCs) of $\mathcal{M}'$, and mark $\mathbf{EG}\neg P$ as true in all states belonging to one of the non-trivial SCCs.

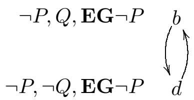
Now, similarly as in the algorithm CheckEU, iteratively collect all the states that are predecessors of at least one state in the non-trivial SCCs of $\mathcal{M}'$. Here the only non-trivial SCC of $\mathcal{M}'$ is $\{b,d\}$. Since the state e is not a predecessor of b or d, the CheckEG algorithm terminates immediately:

Round	Collected states	Considered states	New states
1	$\{b,d\}$	$\{b,d\}$	$\emptyset$

Hence we arrive at



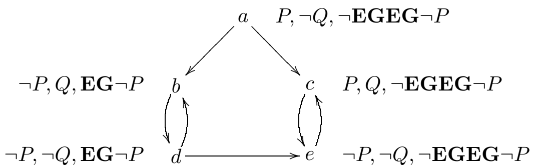
Now determine the states in which the subformula $\mathbf{EGEG}\neg P$ is true using CheckEG. We consider the restriction $\mathcal{M}''$ based on those states in which $\mathbf{EG}\neg P$ is true.



The only non-trivial SCC of $\mathcal{M}''$ is $\{b, d\}$. Again, CheckEG terminates immediately.

Round	Collected states	Considered states	New states
1	$\{b, d\}$	$\{b, d\}$	$\emptyset$

We now know that the formula $\mathbf{EGEG}\neg P$ is true in the states b and d, and hence $\neg\mathbf{EGEG}\neg P$ is true in the states a, c, and e.



Now evaluate the subformula $\mathbf{E}(\top\mathbf{U}P)$ using CheckEU:

Round	Collected states	considered states	New states
1	$\{a, c\}$	$\{a, c\}$	$\{e\}$
2	$\{a, c, e\}$	$\{e\}$	$\{d\}$
3	$\{a, c, d, e\}$	$\{d\}$	$\{b\}$
4	$\{a, b, c, d, e\}$	$\{b\}$	$\emptyset$

Thus, the formula $\mathbf{E}(\top\mathbf{U}P)$ is true in all states of $\mathcal{M}$, and therefore $\neg\mathbf{E}(\top\mathbf{U}P)$ is false in all states. It follows that the conjunction $\neg\mathbf{E}(\top\mathbf{U}P) \wedge \mathbf{EG}\neg P$ is also false in all the states.

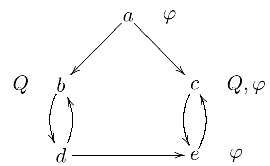
Then we again use CheckEU for considering $\mathbf{E}\big((\mathbf{EG}\neg P)\mathbf{U}(\neg\mathbf{E}(\top\mathbf{U}P) \wedge \mathbf{EG}\neg P)\big)$. The algorithm terminates immediately, since the set of states from which we start from is empty by the above.

Round	Collected states	Considered states	New states
1	$\emptyset$	$\emptyset$	$\emptyset$

Thus the formula $\neg\mathbf{E}\big((\mathbf{EG}\neg P)\mathbf{U}(\neg\mathbf{E}(\top\mathbf{U}P) \wedge \mathbf{EG}\neg P)\big)$ is true in all the states of $\mathcal{M}$. Since we already know that $\neg\mathbf{EGEG}\neg P$ is true in the states a,c, and e, we can evaluate the conjunction

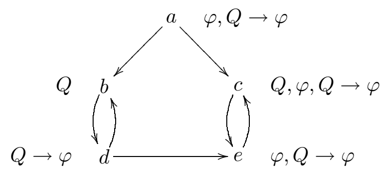
$$\varphi = \neg\mathbf{E}\big((\mathbf{EG}\neg P)\mathbf{U}(\neg\mathbf{E}(\top\mathbf{U}P) \wedge \mathbf{EG}\neg P)\big) \wedge \neg\mathbf{EGEG}\neg P$$

in each state of the model:

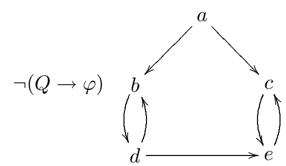


(For clarity, only those subformulas are visible which are still needed.)

Now the subformula $Q \to \varphi$:

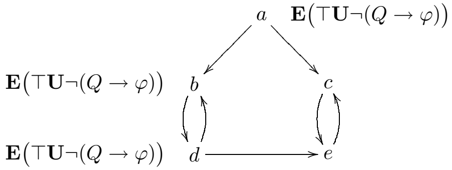


The subformula $\neg(Q \to \varphi)$:

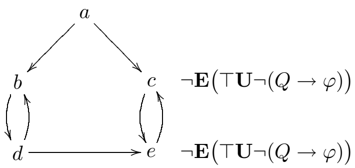


The subformula $\mathbf{E}\big(\top\mathbf{U}\neg(Q \to \varphi)\big)$ using CheckEU:

Round	Collected states	Considered states	New states
1	$\{b\}$	$\{b\}$	$\{a, d\}$
2	$\{a, b, d\}$	$\{a, d\}$	$\emptyset$

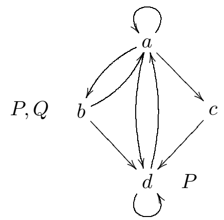


Finally, we arrive at:



T-79.5101 **Spring 2008**
Advanced Course in Computational Logic
Exercise Session 12
Solutions

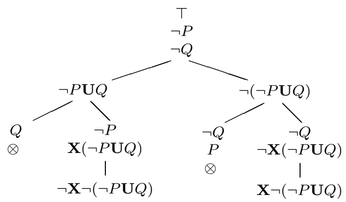
1. $\mathcal{M}$:



The closure of $\mathbf{X}(\neg P\mathbf{U}Q)$:

$$\mathrm{CL}\big(\mathbf{X}(\neg P\mathbf{U}Q)\big) = \big\{\mathbf{X}(\neg P\mathbf{U}Q), \neg\mathbf{X}(\neg P\mathbf{U}Q), \neg P\mathbf{U}Q, \mathbf{X}\neg(\neg P\mathbf{U}Q), \neg(\neg P\mathbf{U}Q), \neg P, Q, \neg\mathbf{X}\neg(\neg P\mathbf{U}Q), P, \neg Q\big\}$$

Now construct the atoms. Since $v(a, P) = v(a, Q) =$ false, we can construct the following atomic tableau:



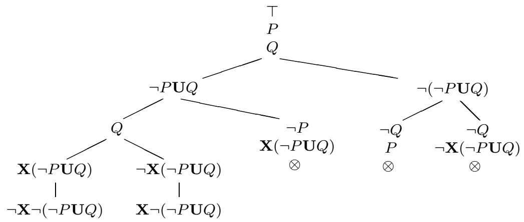
The open branches in the tableau give us the sets of formulas

$$K_1 = \big\{\top, \neg P, \neg Q, \neg P\mathbf{U}Q, \mathbf{X}(\neg P\mathbf{U}Q), \neg\mathbf{X}\neg(\neg P\mathbf{U}Q)\big\}$$
$$K_2 = \big\{\top, \neg P, \neg Q, \neg(\neg P\mathbf{U}Q), \neg\mathbf{X}(\neg P\mathbf{U}Q), \mathbf{X}\neg(\neg P\mathbf{U}Q)\big\}$$

Hence for state a we obtain the atoms (a, K_1) and (a, K_2).

Since the valuation for the formulas P and Q in state c is identical to the valuation in state a, for state c we obtain the atoms (c, K_1) and (c, K_2).

Now consider state b.

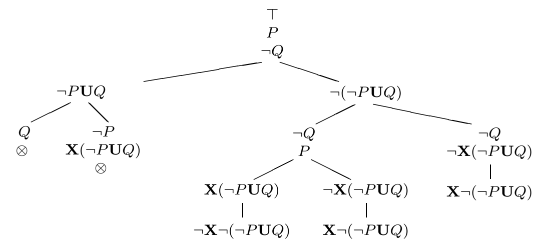


The open branches in the tableau give us the sets of formulas

$$K_3 = \{\top, P, Q, \neg P\mathbf{U}Q, \mathbf{X}(\neg P\mathbf{U}Q), \neg\mathbf{X}\neg(\neg P\mathbf{U}Q)\}$$
$$K_4 = \{\top, P, Q, \neg P\mathbf{U}Q, \neg\mathbf{X}(\neg P\mathbf{U}Q), \mathbf{X}\neg(\neg P\mathbf{U}Q)\}$$

Thus for b we have the atoms (b, K_3) and (b, K_4).

Then consider state d.



Now we arrive at the sets of formulas

$$K_5 = \{\top, P, \neg Q, \neg(\neg P\mathbf{U}Q), \mathbf{X}(\neg P\mathbf{U}Q), \neg\mathbf{X}\neg(\neg P\mathbf{U}Q)\}$$
$$K_6 = \{\top, P, \neg Q, \neg(\neg P\mathbf{U}Q), \neg\mathbf{X}(\neg P\mathbf{U}Q), \mathbf{X}\neg(\neg P\mathbf{U}Q)\}$$

(Two open branches give us K_6!) Thus for d we have the atoms (d, K_5) and (d, K_6).

Then, we construct a graph $G = (N, E)$. The set of nodes N is the set of atoms we have just obtained, that is,

$$N = \{(a, K_1), (a, K_2), (b, K_3), (b, K_4), (c, K_1), (c, K_2), (d, K_5), (d, K_6)\}.$$

The set of edges E is defined as follows: there is an edge from the atom (s, K) to atom (s', K') if and only if

(a) sRs' (in model $\mathcal{M}$), and

(b) for each formula $\mathbf{X}\varphi \in \mathrm{CL}(\mathbf{X}(\neg P\mathbf{U}Q))$ we have $\mathbf{X}\varphi \in K$ if and only if $\varphi \in K'$.

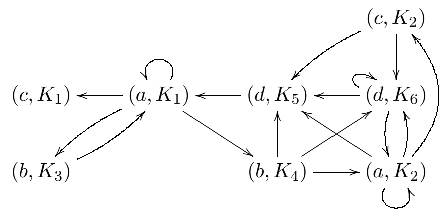
To check condition (b) we can first form a "compatibility relation" over the atoms:

	K_1	K_2	K_3	K_4	K_5	K_6
K_1	×		×	×		
K_2		×			×	×
K_3	×		×	×		
K_4		×			×	×
K_5	×		×	×		
K_6		×			×	×

In the table, there is a tick in the jth column of the ith row if and only if the condition "$\mathbf{X}\varphi \in K_i$ if and only if $\varphi \in K_j$" holds for each $\mathbf{X}\varphi \in \mathrm{CL}(\mathbf{X}(\neg P\mathbf{U}Q))$. For example, this is fulfilled by (K_1, K_3) since $\mathbf{X}(\neg P\mathbf{U}Q) \in K_1$ is the only formula in K_1 of the form $\mathbf{X}\varphi$ and we have $\neg P\mathbf{U}Q \in K_3$.

The conditions (a) and (b) can now be checked using R and this table. As an example, consider the atom (b, K_3). Now any atom in $\{(a, K_1), (a, K_2), (d, K_5), (d, K_6)\}$ together with (b, K_3) fulfills condition (a). However, since condition (b) does not hold for any of (K_3, K_2), (K_3, K_5), (K_3, K_6), the only edge from (b, K_3) is $\langle (b, K_3), (a, K_1)\rangle$.

In the end we arrive at the graph G:



In order to evaluate $\mathbf{EX}(\neg P\mathbf{U}Q)$ in state a, we check whether there is a path x in G fulfilling the following conditions: (i) x begins at one of the atoms (a, K) ($K \in \{K_1, \ldots, K_6\}$), (ii) $\mathbf{X}(\neg P\mathbf{U}Q)$ is in K, and (iii) x leads to a self-fulfilling non-trivial SCC of G.

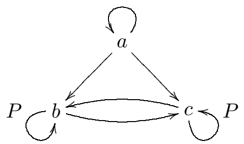
The only non-trivial SCC of G is

$$C = \{(a, K_1), (a, K_2), (b, K_3), (b, K_4), (c, K_2), (d, K_5), (d, K_6)\}.$$

This SCC is also self-fulfilling, since the only formula of the form $\varphi\mathbf{U}\psi$ appearing in the atoms of C is $\neg P\mathbf{U}Q$, and C includes an atom for which $Q \in K_3$ holds ((b, K_3), for example)

Since $(a, K_1) \in C$, the self-fulfilling non-trivial SCC C is reachable from (a, K_1). Thus, since $\mathbf{X}(\neg P\mathbf{U}Q) \in K_1$, we know that $\mathcal{M}, a \models \mathbf{EX}(\neg P\mathbf{U}Q)$ holds.

2. $\mathcal{M}$:



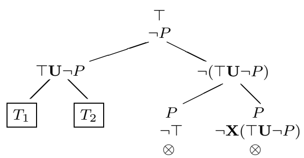
$\mathcal{M}, a \models \mathbf{AFG}P$ iff $\mathcal{M}, a \models \neg\mathbf{E}\neg\mathbf{FG}P$ iff $\mathcal{M}, a \not\models \mathbf{E}\neg\mathbf{FG}P$. Hence we will investigate whether $\mathcal{M}, a \models \mathbf{E}\neg\mathbf{FG}P$ holds. First, rewrite $\neg\mathbf{FG}P$ using only the temporal connectives $\mathbf{X}$ and $\mathbf{U}$.

$$\begin{aligned}\neg\mathbf{FG}P &\equiv \neg\mathbf{F}\neg\mathbf{F}\neg P\\ &\equiv \neg\mathbf{F}\neg(\top\mathbf{U}\neg P)\\ &\equiv \neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big)\end{aligned}$$

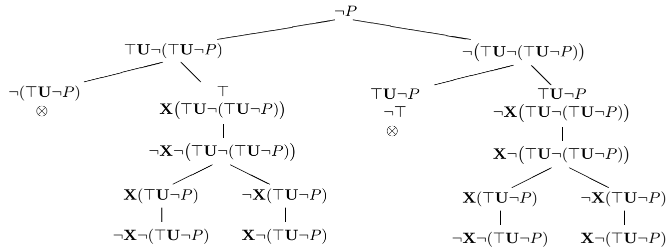
The closure of $\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big)$:

$$\begin{aligned}\mathrm{CL}(\neg(\top\mathbf{U}\neg(\top\mathbf{U}\neg P))) = \{&\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), \top\mathbf{U}\neg(\top\mathbf{U}\neg P),\\ &\top, \neg(\top\mathbf{U}\neg P), \mathbf{X}\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), \neg\top,\\ &\top\mathbf{U}\neg P, \neg\mathbf{X}\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), \neg P,\\ &\mathbf{X}(\top\mathbf{U}\neg P), \mathbf{X}\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), P,\\ &\neg\mathbf{X}(\top\mathbf{U}\neg P), \neg\mathbf{X}\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big),\\ &\mathbf{X}\neg(\top\mathbf{U}\neg P), \neg\mathbf{X}\neg(\top\mathbf{U}\neg P)\}\end{aligned}$$

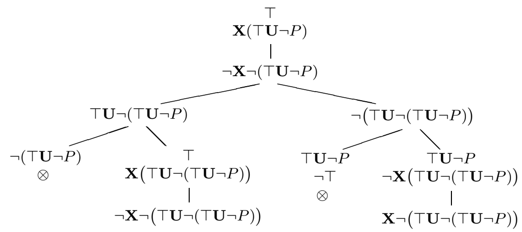
Construct the atoms. Since $v(a, P) = \text{false}$, for state a we can construct the atomic tableau



where the branch T_1 is



and the branch T_2



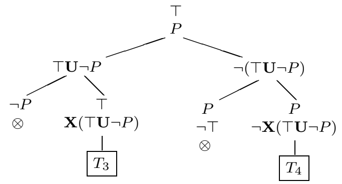
From the open branches we obtain

$$\begin{aligned}K_1 = \{&\top, \neg P, \top\mathbf{U}\neg P, \top\mathbf{U}\neg(\top\mathbf{U}\neg P), \mathbf{X}\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big),\\ &\neg\mathbf{X}\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), \mathbf{X}(\top\mathbf{U}\neg P), \neg\mathbf{X}\neg(\top\mathbf{U}\neg P)\}\\ K_2 = \{&\top, \neg P, \top\mathbf{U}\neg P, \top\mathbf{U}\neg(\top\mathbf{U}\neg P), \mathbf{X}\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big),\\ &\neg\mathbf{X}\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), \neg\mathbf{X}(\top\mathbf{U}\neg P), \mathbf{X}\neg(\top\mathbf{U}\neg P)\}\\ K_3 = \{&\top, \neg P, \top\mathbf{U}\neg P, \neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), \neg\mathbf{X}\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big),\\ &\mathbf{X}\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), \mathbf{X}(\top\mathbf{U}\neg P), \neg\mathbf{X}\neg(\top\mathbf{U}\neg P)\}\\ K_4 = \{&\top, \neg P, \top\mathbf{U}\neg P, \neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), \neg\mathbf{X}\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big),\\ &\mathbf{X}\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), \neg\mathbf{X}(\top\mathbf{U}\neg P), \mathbf{X}\neg(\top\mathbf{U}\neg P)\}\end{aligned}$$

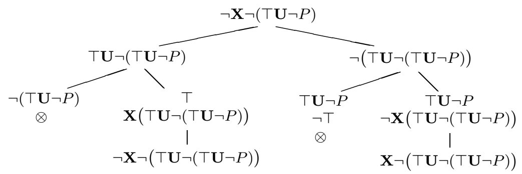
$$\begin{aligned}K_5 = \{&\top, \neg P, \top\mathbf{U}\neg P, \mathbf{X}(\top\mathbf{U}\neg P), \neg\mathbf{X}\neg(\top\mathbf{U}\neg P), \top\mathbf{U}\neg(\top\mathbf{U}\neg P),\\ &\mathbf{X}\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), \neg\mathbf{X}\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big)\}\\ K_6 = \{&\top, \neg P, \top\mathbf{U}\neg P, \mathbf{X}(\top\mathbf{U}\neg P), \neg\mathbf{X}\neg(\top\mathbf{U}\neg P), \neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big),\\ &\neg\mathbf{X}\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), \mathbf{X}\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big)\}\end{aligned}$$

Thus we obtain the atoms (a, K_1), (a, K_2), (a, K_3), (a, K_4), (a, K_5), and (a, K_6).

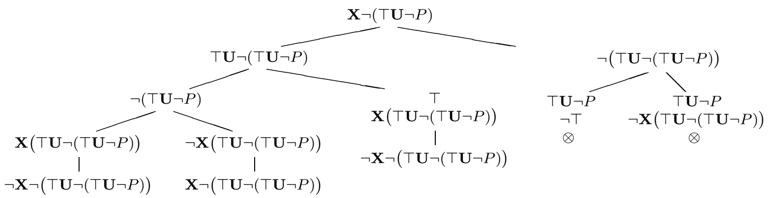
Since $v(b, P) = v(c, P) =$ true, for b and c we can construct the atomic tableau



where the branch T_3 is



and the branch T_4



From the open branches we obtain

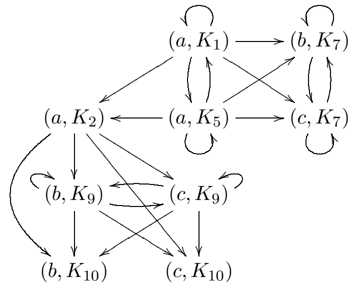
$$\begin{aligned}
K_7 &= \big\{\top, P, \top\mathbf{U}\neg P, \mathbf{X}(\top\mathbf{U}\neg P), \neg\mathbf{X}\neg(\top\mathbf{U}\neg P), \top\mathbf{U}\neg(\top\mathbf{U}\neg P),\\
&\quad \mathbf{X}\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), \neg\mathbf{X}\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big)\big\}\\
K_8 &= \big\{\top, P, \top\mathbf{U}\neg P, \mathbf{X}(\top\mathbf{U}\neg P), \neg\mathbf{X}\neg(\top\mathbf{U}\neg P), \neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big),\\
&\quad \neg\mathbf{X}\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), \mathbf{X}\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big)\big\}\\
K_9 &= \big\{\top, P, \neg(\top\mathbf{U}\neg P), \neg\mathbf{X}(\top\mathbf{U}\neg P), \mathbf{X}\neg(\top\mathbf{U}\neg P), \top\mathbf{U}\neg(\top\mathbf{U}\neg P),\\
&\quad \mathbf{X}\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), \neg\mathbf{X}\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big)\big\}\\
K_{10} &= \big\{\top, P, \neg(\top\mathbf{U}\neg P), \neg\mathbf{X}(\top\mathbf{U}\neg P), \mathbf{X}\neg(\top\mathbf{U}\neg P), \top\mathbf{U}\neg(\top\mathbf{U}\neg P),\\
&\quad \neg\mathbf{X}\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big), \mathbf{X}\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big)\big\}
\end{aligned}$$

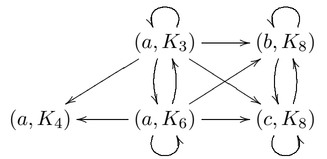
Thus we obtain the atoms (b, K_7), (b, K_8), (b, K_9), (b, K_{10}), (c, K_7), (c, K_8), (c, K_9), and (c, K_{10}).

Now we have the following "compatibility relation":

	K_1	K_2	K_3	K_4	K_5	K_6	K_7	K_8	K_9	K_{10}
K_1	×	×			×		×			
K_2									×	×
K_3			×	×		×		×		
K_4										
K_5	×	×			×		×			
K_6			×	×		×		×		
K_7	×	×			×		×			
K_8			×	×		×		×		
K_9									×	×
K_{10}										

The graph G:





The non-trivial SCCs of G:

$$\begin{aligned}
C_1 &= \big\{(a, K_1), (a, K_5)\big\}\\
C_2 &= \big\{(a, K_3), (a, K_6)\big\}\\
C_3 &= \big\{(b, K_7), (c, K_7)\big\}\\
C_4 &= \big\{(b, K_8), (c, K_8)\big\}\\
C_5 &= \big\{(b, K_9), (c, K_9)\big\}
\end{aligned}$$

Out of these, C_2 and C_5 are self-fulfilling. Now check whether C_2 or C_5 is reachable from some atom (a, K), where $\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big) \in K$. Since the formula $\neg\big(\top\mathbf{U}\neg(\top\mathbf{U}\neg P)\big)$ is in K_6 and C_2 is reachable from

(a, K_6) (since $(a, K_6) \in C_2$), it follows that $\mathcal{M}, a \models \mathbf{E}\neg\mathbf{FG}P$ holds. Thus $\mathcal{M}, a \models \mathbf{AFG}P$ *does not hold.*

T-79.5101 **Spring 2008**
Advanced Course in Computational Logic
Exercise Session 13
Solutions

1. We start from the negation of the given formula,

$$\neg\Big(\big(Q \vee (P \wedge \mathbf{AXA}(P\mathbf{U}Q))\big) \rightarrow \mathbf{A}(P\mathbf{U}Q)\Big),$$

and translate it to positive normal form:

$$\big(Q \vee (P \wedge \mathbf{AXA}(P\mathbf{U}Q))\big) \wedge \neg\mathbf{A}(P\mathbf{U}Q)$$
$$\big(Q \vee (P \wedge \mathbf{AXA}(P\mathbf{U}Q))\big) \wedge \mathbf{E}(\neg P\mathbf{B}Q)$$

Now construct the CTL tableau. We start with the OR-node

$$D_0 = \Big\{\big(Q \vee (P \wedge \mathbf{AXA}(P\mathbf{U}Q))\big) \wedge \mathbf{E}(\neg P\mathbf{B}Q)\Big\},$$

the AND-successors of which are

$$\begin{aligned}
C_0 = \Big\{&\big(Q \vee (P \wedge \mathbf{AXA}(P\mathbf{U}Q))\big) \wedge \mathbf{E}(\neg P\mathbf{B}Q),\\
&Q \vee \big(P \wedge \mathbf{AXA}(P\mathbf{U}Q)\big), \mathbf{E}(\neg P\mathbf{B}Q),\\
&Q, \neg Q, \neg P \vee \mathbf{EXE}(\neg P\mathbf{B}Q), \neg P\Big\}\\
C_1 = \Big\{&\big(Q \vee (P \wedge \mathbf{AXA}(P\mathbf{U}Q))\big) \wedge \mathbf{E}(\neg P\mathbf{B}Q),\\
&Q \vee \big(P \wedge \mathbf{AXA}(P\mathbf{U}Q)\big), \mathbf{E}(\neg P\mathbf{B}Q),\\
&Q, \neg Q, \neg P \vee \mathbf{EXE}(\neg P\mathbf{B}Q), \mathbf{EXE}(\neg P\mathbf{B}Q)\Big\}\\
C_2 = \Big\{&\big(Q \vee (P \wedge \mathbf{AXA}(P\mathbf{U}Q))\big) \wedge \mathbf{E}(\neg P\mathbf{B}Q),\\
&Q \vee \big(P \wedge \mathbf{AXA}(P\mathbf{U}Q)\big), \mathbf{E}(\neg P\mathbf{B}Q),\\
&P \wedge \mathbf{AXA}(P\mathbf{U}Q), P, \mathbf{AXA}(P\mathbf{U}Q), \neg Q,\\
&\neg P \vee \mathbf{EXE}(\neg P\mathbf{B}Q), \neg P\Big\}\\
C_3 = \Big\{&\big(Q \vee (P \wedge \mathbf{AXA}(P\mathbf{U}Q))\big) \wedge \mathbf{E}(\neg P\mathbf{B}Q),\\
&Q \vee \big(P \wedge \mathbf{AXA}(P\mathbf{U}Q)\big), \mathbf{E}(\neg P\mathbf{B}Q),\\
&P \wedge \mathbf{AXA}(P\mathbf{U}Q), P, \mathbf{AXA}(P\mathbf{U}Q), \neg Q,\\
&\neg P \vee \mathbf{EXE}(\neg P\mathbf{B}Q), \mathbf{EXE}(\neg P\mathbf{B}Q)\Big\}
\end{aligned}$$

The nodes C_0, C_1, and C_2 can be pruned, since they are contradictory (contain the formulas ϕ and $\neg\phi$ for some ϕ). (In other words, we can in fact prune contradictory AND-nodes already when constructing AND-nodes without affecting the end result! Left as an exercise.)

Since the node C_3 contains the formulas $\mathbf{AXA}(P\mathbf{U}Q)$ and $\mathbf{EXE}(\neg P\mathbf{B}Q)$, for C_3 we have the OR-successor $D_1 = \{\mathbf{A}(P\mathbf{U}Q), \mathbf{E}(\neg P\mathbf{B}Q)\}$.

The AND-successors of D_1:

$$\begin{aligned}
C_4 &= D_1 \cup \{Q, \neg Q, \neg P \vee \mathbf{EXE}(\neg P\mathbf{B}Q), \neg P\} \\
C_5 &= D_1 \cup \{Q, \neg Q, \neg P \vee \mathbf{EXE}(\neg P\mathbf{B}Q), \mathbf{EXE}(\neg P\mathbf{B}Q)\} \\
C_6 &= D_1 \cup \{P \wedge \mathbf{AXA}(P\mathbf{U}Q), P, \mathbf{AXA}(P\mathbf{U}Q), \neg Q, \\
&\qquad \neg P \vee \mathbf{EXE}(\neg P\mathbf{B}Q), \neg P\} \\
C_7 &= D_1 \cup \{P \wedge \mathbf{AXA}(P\mathbf{U}Q), P, \mathbf{AXA}(P\mathbf{U}Q), \neg Q, \\
&\qquad \neg P \vee \mathbf{EXE}(\neg P\mathbf{B}Q), \mathbf{EXE}(\neg P\mathbf{B}Q)\}
\end{aligned}$$

The nodes C_4, C_5, and C_6 are contradictory and hence pruned. We are left with C_7 for which we obtain the OR-successor $\{\mathbf{A}(P\mathbf{U}Q), \mathbf{E}(\neg P\mathbf{B}Q)\} = D_1$.

The AND-node C_7 is pruned since it contains the eventuality formula $\mathbf{A}(P\mathbf{U}Q)$ which is not satisfiable. (Since C_7 does not contain Q, we have that D_1 must be in the required acyclic subgraph that would imply that $\mathbf{A}(P\mathbf{U}Q)$ is satisfiable. However, since C_7 is the only successor of D_1, there is no such acyclic subgraph.)

After removing C_7 we can step-by-step remove the nodes D_1, C_3, and D_0. Since the resulting tableau does not contain an AND-node which contains the formula

$$\Big(Q \vee \big(P \wedge \mathbf{AXA}(P\mathbf{U}Q)\big)\Big) \wedge \mathbf{E}(\neg P\mathbf{B}Q),$$

it follows that the formula is unsatisfiable. Hence the negation of this formula (the original formula given in the assignment) is valid.

2. Determine the positive normal form:

$$\begin{gathered}
\mathbf{GF}P \rightarrow \mathbf{GF}\neg P \\
\neg\mathbf{GF}P \vee \mathbf{GF}\neg P \\
\mathbf{FG}\neg P \vee \mathbf{GF}\neg P
\end{gathered}$$

Then, replace the LTL connectives $\mathbf{F}$ and $\mathbf{G}$ with the CTL connective $\mathbf{AF}$ and $\mathbf{AG}$, respectively. We obtain the formula

$$\mathbf{AFAG}\neg P \vee \mathbf{AGAF}\neg P.$$

This CTL formula is satisfiable if and only if the original LTL formula is satisfiable. We can thus apply the CTL tableau method. The root of the tableau is the OR-node

$$D_0 = \{\mathbf{AFAG}\neg P \vee \mathbf{AGAF}\neg P\}.$$

The AND-successors of D_0:

$$\begin{aligned}
C_0 &= \{\mathbf{AFAG}\neg P \vee \mathbf{AGAF}\neg P, \mathbf{AFAG}\neg P, \mathbf{AG}\neg P, \neg P, \mathbf{AXAG}\neg P\} \\
C_1 &= \{\mathbf{AFAG}\neg P \vee \mathbf{AGAF}\neg P, \mathbf{AFAG}\neg P, \mathbf{AXAFAG}\neg P\} \\
C_2 &= \{\mathbf{AFAG}\neg P \vee \mathbf{AGAF}\neg P, \mathbf{AGAF}\neg P, \mathbf{AF}\neg P, \\
&\qquad \mathbf{AXAGAF}\neg P, \neg P\} \\
C_3 &= \{\mathbf{AFAG}\neg P \vee \mathbf{AGAF}\neg P, \mathbf{AGAF}\neg P, \mathbf{AF}\neg P, \\
&\qquad \mathbf{AXAGAF}\neg P, \mathbf{AXAF}\neg P\}
\end{aligned}$$

The OR-successor of C_0: $D_1 = \{\mathbf{AG}\neg P\}$
The OR-successor of C_1: $D_2 = \{\mathbf{AFAG}\neg P\}$
The OR-successor of C_2: $D_3 = \{\mathbf{AGAF}\neg P\}$
The OR-successor of C_3: $D_4 = \{\mathbf{AGAF}\neg P, \mathbf{AF}\neg P\}$

The AND-successor of D_1: $C_4 = \{\mathbf{AG}\neg P, \neg P, \mathbf{AXAG}\neg P\}$

The AND-successors of D_2:

$$\begin{aligned}
C_5 &= \{\mathbf{AFAG}\neg P, \mathbf{AG}\neg P, \neg P, \mathbf{AXAG}\neg P\} \\
C_6 &= \{\mathbf{AFAG}\neg P, \mathbf{AXAFAG}\neg P\}
\end{aligned}$$

The AND-successors of D_3:

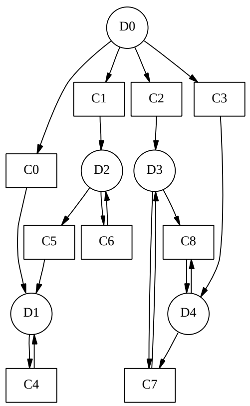
$$\begin{aligned}
C_7 &= \{\mathbf{AGAF}\neg P, \mathbf{AF}\neg P, \mathbf{AXAGAF}\neg P, \neg P\} \\
C_8 &= \{\mathbf{AGAF}\neg P, \mathbf{AF}\neg P, \mathbf{AXAGAF}\neg P, \mathbf{AXAF}\neg P\}
\end{aligned}$$

The AND-successors of D_4:

$$\begin{aligned}
C_9 &= \{\mathbf{AGAF}\neg P, \mathbf{AF}\neg P, \mathbf{AXAGAF}\neg P, \neg P\} = C_7 \\
C_{10} &= \{\mathbf{AGAF}\neg P, \mathbf{AF}\neg P, \mathbf{AXAGAF}\neg P, \mathbf{AXAF}\neg P\} = C_8
\end{aligned}$$

The OR-successor of C_4: $D_5 = \{\mathbf{AG}\neg P\} = D_1$
The OR-successor of C_5: $D_6 = \{\mathbf{AG}\neg P\} = D_1$
The OR-successor of C_6: $D_7 = \{\mathbf{AFAG}\neg P\} = D_2$
The OR-successor of C_7: $D_8 = \{\mathbf{AGAF}\neg P\} = D_3$
The OR-successor of C_8: $D_9 = \{\mathbf{AGAF}\neg P, \mathbf{AF}\neg P\} = D_4$

The initial tableau T_0:



Since the node C_4 is not contradictory and does not contain any eventuality formulas, it remains in the final tableau. Thus the OR-node D_1 will have a successor, and hence D_1 will remain as well. All successors of node C_0 will then also remain. Furthermore, node C_0 remains since it is not contradictory and all of its eventuality formulas (there is only one, $\mathbf{AFAG}\neg P$) are satisfiable in the initial tableau.

It follows that the final tableau obtained from T_0 contains the AND-node C_0 which contains the formula $\mathbf{AFAG}\neg P \lor \mathbf{AGAF}\neg P$. Hence this CTL formula is satisfiable.

Using the nodes C_0 and C_4 we can now construct a model for the CTL formula:

$$\begin{tikzcd} C_0 \arrow[r] & C_4 \arrow[loop right] \\ \neg P & \neg P \end{tikzcd}$$

Since the CTL formula $\mathbf{AFAG}\neg P \lor \mathbf{AGAF}\neg P$ is satisfiable, the original LTL formula $\mathbf{FG}\neg P \lor \mathbf{GF}\neg P \equiv \mathbf{GF}P \to \mathbf{GF}\neg P$ is also satisfiable.