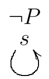
**T-79.5101** **Spring 2008**
**Advanced Course in Computational Logic**
**Exercise Session 4**
**Solutions**

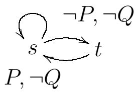
1. One possible counterexample is the model $\mathcal{M} = \langle S, R, v\rangle$, where $S = \{s\}$, $R = \{\langle s, s\rangle\}$, and $v(s, P) =$ false.




$\mathcal{M} \models \Box P \to \Diamond P$ holds (since $\mathcal{M}, s \not\Vdash \Box P$), and $\mathcal{M}, s \Vdash \Box\neg P$ holds since $\langle s, s\rangle \in R$, $\mathcal{M}, s \Vdash \neg P$ and $s$ has no other successors. Furthermore, $\mathcal{M}, s \Vdash \Box\Box\neg P$ holds. On the other hand, $\mathcal{M}, s \not\Vdash \Box P$ (and $s$ has no other successor), and hence $\mathcal{M}, s \Vdash \Diamond\Box P$ does not holds. Thus $\mathcal{M}$ is a counterexample.

(Notice that counterexamples are not unique in general: here other possibilities for counterexamples are, for examples, $\mathcal{M}' = \langle S', R', v'\rangle$, where $S' = \{s', t'\}$, $R' = \{\langle s', t'\rangle, \langle t', s'\rangle\}$ and $v(s', P) = v(t', P) =$ false, and $\mathcal{M}'' = \langle S'', R'', v''\rangle$, $S'' = \{s'', t'', u''\}$, $R'' = \{\langle s'', t''\rangle, \langle t'', u''\rangle, \langle u'', t''\rangle\}$ and $v''(s'', P) = v''(t'', P) =$ true, $v''(u'', P) =$ false, considering the worlds $s'$ and $s''$, respectively.)

2. $\mathcal{M} = \langle S, R, v\rangle$, where $S = \{s, t\}$, $R = \{\langle s, s\rangle, \langle s, t\rangle, \langle t, s\rangle\}$, $v(s, P) =$ true and $v(s, Q) = v(t, P) = v(t, Q) =$ false.




$\mathcal{M}, s \Vdash \Diamond P \vee \Diamond Q$ and $\mathcal{M}, t \Vdash \Diamond P \vee \Diamond Q$ hold (since $\mathcal{M}, s \Vdash P$, $\langle s, s\rangle \in R$ and $\langle t, s\rangle \in R$), and $\mathcal{M}, s \Vdash \neg\Box P$ holds, since $\langle s, t\rangle \in R$ and $\mathcal{M}, t \not\Vdash P$. However, $\mathcal{M}, s \not\Vdash \Diamond Q$, since $\mathcal{M}, s' \not\Vdash Q$ for all $s' \in S$ for which $\langle s, s'\rangle \in R$. Thus $\mathcal{M}$ is a counterexample.

3. Assume that
$$\Sigma \cup \{P\} \not\models_{\mathbf{L}} \Upsilon \Longrightarrow Q.$$
Then there is a model $\mathcal{M} = \langle S, R, v\rangle$ such that
$$\mathcal{M} \models \Sigma \cup \{P\}$$



and
$$\exists s \in S : \quad \forall \varphi \in \Upsilon : \mathcal{M}, s \Vdash \varphi, \text{ but } \mathcal{M}, s \not\Vdash Q.$$
Especially, $\mathcal{M}, t \Vdash P$ for all $t \in S$, and hence
$$\mathcal{M}, s \Vdash P \wedge \Box P \wedge \Box\Box P \wedge \Box\Box\Box P.$$
Since additionally $\mathcal{M} \models \Sigma$ holds we have
$$\Sigma \not\models_{\mathbf{L}} \Upsilon \Longrightarrow P \wedge \Box P \wedge \Box\Box P \wedge \Box\Box\Box P \to Q.$$

4. a) Assume that the frame $\mathcal{F} = \langle S, R\rangle$ is transitive and that the formula $\Box P \to \Box\Box P$ is not valid in the frame. Then there is a model $\mathcal{M} = \langle S, R, v\rangle$ (based on $\mathcal{F}$) and a world $s \in S$ such that $\mathcal{M}, s \not\Vdash \Box P \to \Box\Box P$. Now, $\mathcal{M}, s \Vdash \Box P$ but on the other hand $\mathcal{M}, s \not\Vdash \Box\Box P$. From the latter it follows that there is a world $t \in S$ for which $\langle s, t\rangle \in R$ and $\mathcal{M}, t \not\Vdash \Box P$. Furthermore, there is a world $u \in S$ for which $\langle t, u\rangle \in R$ and $\mathcal{M}, u \not\Vdash P$. Since $\langle s, t\rangle \in R$ and $\langle t, u\rangle \in R$ by transitivity of $\mathcal{F}$ we have $\langle s, u\rangle \in R$. Since $\langle s, u\rangle \in R$ and $\mathcal{M}, u \not\Vdash P$, we have $\mathcal{M}, s \not\Vdash \Box P$, which is in contradiction with the assumption that $\mathcal{M}, s \Vdash \Box P$ holds. Thus the formula $\Box P \to \Box\Box P$ is valid in $\mathcal{F}$.

   b) Assume that the frame $\mathcal{F} = \langle S, R\rangle$ is euclidean. Take an arbitrary model $\mathcal{M} = \langle S, R, v\rangle$ based on $\mathcal{F}$ and an arbitrary world $s \in S$ for which $\mathcal{M}, s \Vdash \neg\Box P$ holds. Then $\mathcal{M}, s \not\Vdash \Box P$, and hence there is a world $t \in S$ such that $\langle s, t\rangle \in R$ and $\mathcal{M}, t \not\Vdash P$. Assume $\langle s, u\rangle \in R$. Since $\langle s, t\rangle \in R$ and the frame is euclidean, we have $\langle u, t\rangle \in R$. Hence $\mathcal{M}, u \not\Vdash \Box P$ and $\mathcal{M}, u \Vdash \neg\Box P$. Since $u$ is an arbitrary successor of $s$, we have $\mathcal{M}, s \Vdash \Box\neg\Box P$, and hence
$$\mathcal{M}, s \Vdash \neg\Box P \to \Box\neg\Box P.$$
Thus $\neg\Box P \to \Box\neg\Box P$ is valid in $\mathcal{M}$, and $\neg\Box P \to \Box\neg\Box P$ is valid in $\mathcal{F}$ (since $\mathcal{M}$ was chosen arbitrarily).

5. Assume that $\mathcal{F} = \langle S, R\rangle$ is reflexive and euclidean. If $sRt$, by reflexivity we have $sRs$. Since the frame is also euclidean, we have $tRs$, and thus the frame is symmetric.

   Now assume $sRt$ and $tRu$. By symmetricity we have $tRs$. Since the frame is also euclidean, we also have $sRu$. Thus the frame is transitive.

**T-79.5101** **Spring 2008**
**Advanced Course in Computational Logic**
**Exercise Session 5**
**Solutions**

Axiom $K$:

$$\text{K:} \quad \Box(P \to Q) \to (\Box P \to \Box Q)$$

Inference rules:

$$\text{MP:} \quad \frac{P, P \to Q}{Q}$$

$$\text{N:} \quad \frac{P}{\Box P}$$

1. a) Assume that the frame $\mathcal{F} = \langle S, R\rangle$ is serial and that the formula $\Box P \to \Diamond P$ is not valid under $\mathcal{F}$. Then there is a model $\mathcal{M} = \langle S, R, v\rangle$ based on $\mathcal{F}$ and a world $s \in S$ such that $\mathcal{M}, s \not\Vdash \Box P \to \Diamond P$ holds. Hence, $\mathcal{M}, s \Vdash \Box P$ and $\mathcal{M}, s \not\Vdash \Diamond P$. From $\mathcal{M}, s \not\Vdash \Diamond P$ it follows that there is no world $t \in S$ such that $\langle s, t\rangle \in R$ and $\mathcal{M}, t \Vdash P$. Furthermore, $\mathcal{F}$ is serial by assumption, and thus there is a world $t \in S$ such that $\langle s, t\rangle \in R$. Hence $\mathcal{M}, s \not\Vdash \Box P$. A contradiction follows since $\mathcal{M}, s \Vdash \Box P$, and thus the formula $\Box P \to \Diamond P$ is valid under $\mathcal{F}$.

   b) Assume that the frame $\mathcal{F} = \langle S, R\rangle$ is weakly dence and that the formula $\Box\Box P \to \Box P$ is not valid under $\mathcal{F}$. Then there is a model $\mathcal{M} = \langle S, R, v\rangle$ based on $\mathcal{F}$ and a world $s \in S$ such that $\mathcal{M}, s \not\Vdash \Box\Box P \to \Box P$ holds. Hence $\mathcal{M}, s \Vdash \Box\Box P$ and $\mathcal{M}, s \not\Vdash \Box P$. From $\mathcal{M}, s \not\Vdash \Box P$ it follows that there is a world $t \in S$ such that $\langle s, t\rangle \in R$ and $\mathcal{M}, t \not\Vdash P$. The frame $\mathcal{F}$ is weakly dence by assumption, and thus there is a $u \in S$ such that $\langle s, u\rangle \in R$ and $\langle u, t\rangle \in R$. Since $\langle u, t\rangle \in R$ and $\mathcal{M}, t \not\Vdash P$, it follows that $\mathcal{M}, u \not\Vdash \Box P$. Now $\langle s, u\rangle \in R$ and $\mathcal{M}, u \not\Vdash \Box P$, so $\mathcal{M}, s \not\Vdash \Box\Box P$ must hold. A contradiction follows since $\mathcal{M}, s \Vdash \Box\Box P$, and thus the formula $\Box\Box P \to \Box P$ is valid under $\mathcal{F}$.



2. a)

| | | |
|---|---|---|
| 1. | $P \to (Q \to P)$ | [Tautology] |
| 2. | $\Box\big(P \to (Q \to P)\big)$ | [N, 1] |
| 3. | $\Box\big(P \to (Q \to P)\big) \to \big(\Box P \to \Box(Q \to P)\big)$ | [K] |
| 4. | $\Box P \to \Box(Q \to P)$ | [MP, 2, 3] |

   b)

| | | |
|---|---|---|
| 1. | $\Box(P \to Q)$ | [GP] |
| 2. | $(P \to Q) \to (\neg Q \to \neg P)$ | [Tautology] |
| 3. | $\Box\big((P \to Q) \to (\neg Q \to \neg P)\big)$ | [N, 2] |
| 4. | $\Box\big((P \to Q) \to (\neg Q \to \neg P)\big) \to$ $\big(\Box(P \to Q) \to \Box(\neg Q \to \neg P)\big)$ | [K] |
| 5. | $\Box(P \to Q) \to \Box(\neg Q \to \neg P)$ | [MP, 3, 4] |
| 6. | $\Box(\neg Q \to \neg P)$ | [MP, 1, 5] |
| 7. | $\Box(\neg Q \to \neg P) \to (\Box\neg Q \to \Box\neg P)$ | [K] |
| 8. | $\Box\neg Q \to \Box\neg P$ | [MP, 6, 7] |

3. a)

| | | |
|---|---|---|
| 1. | $P \to Q$ | [GP] |
| 2. | $\neg Q \to P$ | [GP] |
| 3. | $(P \to Q) \to \big((\neg Q \to P) \to Q\big)$ | [Tautology] |
| 4. | $(\neg Q \to P) \to Q$ | [MP, 1, 3] |
| 5. | $Q$ | [MP, 2, 4] |
| 6. | $\Box Q$ | [N, 5] |
| 7. | $\neg Q \lor S$ | [LP] |
| 8. | $(\neg Q \lor S) \to (Q \to S)$ | [Tautology] |
| 9. | $Q \to S$ | [MP, 7, 8] |
| 10. | $S$ | [MP, 5, 9] |
| 11. | $\Box Q \to (S \to \Box Q \land S)$ | [Tautology] |
| 12. | $S \to \Box Q \land S$ | [MP, 6, 11] |
| 13. | $\Box Q \land S$ | [MP, 10, 12] |

b)

| | | |
|---|---|---|
| 1. | $Q \to \neg P$ | [GP] |
| 2. | $\Box(Q \to \neg P)$ | [N, 1] |
| 3. | $\Box(Q \to \neg P) \to (\Box Q \to \Box\neg P)$ | [K] |
| 4. | $\Box Q \to \Box\neg P$ | [MP, 2, 3] |
| 5. | $\Diamond Q \to \Box Q$ | [GP] |
| 6. | $(\Diamond Q \to \Box Q) \to$ $\big((\Box Q \to \Box\neg P) \to (\Diamond Q \to \Box\neg P)\big)$ | [Tautology] |
| 7. | $(\Box Q \to \Box\neg P) \to (\Diamond Q \to \Box\neg P)$ | [MP, 5, 6] |
| 8. | $\Diamond Q \to \Box\neg P$ | [MP, 4, 7] |
| 9. | $(\neg\Box\neg Q \to \Box\neg P) \to (\neg\Box\neg P \to \Box\neg Q)$ | [Tautology] |
| 10. | $\neg\Box\neg P \to \Box\neg Q$ | [MP, 8, 9] |
| 11. | $\Diamond P$ | [LP] |
| 12. | $\Box\neg Q$ | [MP, 10, 11] |



**T-79.5101** **Spring 2008**
**Advanced Course in Computational Logic**
**Exercise Session 6**
**Solutions**

1. a)

| | | |
|---|---|---|
| 1. | $\langle 1\rangle\neg\big(\Box P \to \Box(Q \to P)\big)$ | |
| 2. | $\langle 1\rangle\Box P$ | (1) |
| 3. | $\langle 1\rangle\neg\Box(Q \to P)$ | (1) |
| 4. | $\langle 1,2\rangle\neg(Q \to P)$ | (3) |
| 5. | $\langle 1,2\rangle Q$ | (4) |
| 6. | $\langle 1,2\rangle\neg P$ | (4) |
| 7. | $\langle 1,2\rangle P$ | (2) |
| | $\otimes$ | |

b)

| | | |
|---|---|---|
| 1. | $\langle 1\rangle\neg\big(\Box(P \to Q) \to (\neg\Box\neg P \to \neg\Box\neg Q)\big)$ | |
| 2. | $\langle 1\rangle\Box(P \to Q)$ | (1) |
| 3. | $\langle 1\rangle\neg(\neg\Box\neg P \to \neg\Box\neg Q)$ | (1) |
| 4. | $\langle 1\rangle\neg\Box\neg P$ | (3) |
| 5. | $\langle 1\rangle\neg\neg\Box\neg Q$ | (3) |
| 6. | $\langle 1\rangle\Box\neg Q$ | (5) |
| 7. | $\langle 1,2\rangle\neg\neg P$ | (4) |
| 8. | $\langle 1,2\rangle P$ | (7) |
| 9. | $\langle 1,2\rangle\neg Q$ | (6) |
| 10. | $\langle 1,2\rangle P \to Q$ | (2) |
| 11. | $\langle 1,2\rangle\neg P$ (10) $\otimes$ | 12. $\langle 1,2\rangle Q$ (10) $\otimes$ |

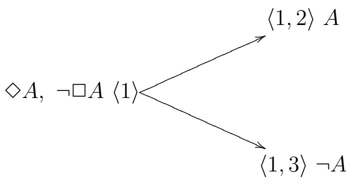
c)

| | | |
|---|---|---|
| 1. | $\langle 1\rangle\neg\big((\Box P \wedge \Box Q) \to \Box(P \wedge Q)\big)$ | |
| 2. | $\langle 1\rangle\Box P \wedge \Box Q$ | (1) |
| 3. | $\langle 1\rangle\neg\Box(P \wedge Q)$ | (1) |
| 4. | $\langle 1\rangle\Box P$ | (2) |
| 5. | $\langle 1\rangle\Box Q$ | (2) |
| 6. | $\langle 1,2\rangle\neg(P \wedge Q)$ | (3) |
| 7. | $\langle 1,2\rangle P$ | (4) |
| 8. | $\langle 1,2\rangle Q$ | (5) |
| 9. | $\langle 1,2\rangle\neg P$ (6) $\otimes$ | 10. $\langle 1,2\rangle\neg Q$ (6) $\otimes$ |

2. a)

| | | |
|---|---|---|
| 1. | $\langle 1\rangle\neg(\Diamond A \to \Box A)$ | |
| 2. | $\langle 1\rangle\Diamond A$ | (1) |
| 3. | $\langle 1\rangle\neg\Box A$ | (1) |
| 4. | $\langle 1,2\rangle\neg\neg A$ | (2; $\Diamond$ stands for $\neg\Box\neg$) |
| 5. | $\langle 1,2\rangle A$ | (4) |
| 6. | $\langle 1,3\rangle\neg A$ | (3) |

Not **K**-valid.




b)

| | | |
|---|---|---|
| 1. | $\langle 1\rangle\neg(\Diamond\Box A \vee \Box\Diamond\neg A)$ | |
| 2. | $\langle 1\rangle\neg\Diamond\Box A$ | (1) |
| 3. | $\langle 1\rangle\neg\Box\Diamond\neg A$ | (1) |
| 4. | $\langle 1\rangle\Box\neg\Box A$ | (2) |
| 5. | $\langle 1,2\rangle\neg\Diamond\neg A$ | (3) |
| 6. | $\langle 1,2\rangle\neg\Box A$ | (4) |
| 7. | $\langle 1,2\rangle\Box\neg\neg A$ | (5) |
| 8. | $\langle 1,2,3\rangle\neg A$ | (6) |
| 9. | $\langle 1,2,3\rangle\neg\neg A$ | (7) |
| | $\otimes$ | |

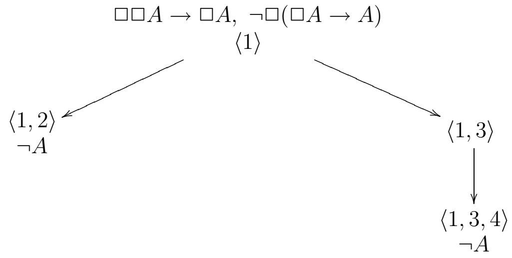
The formula is **K**-valid.

c)

| | | | | | |
|---|---|---|---|---|---|
| 1. | $\langle 1\rangle\neg\big((\Box\Box A \to \Box A) \to \Box(\Box A \to A)\big)$ | | | | |
| 2. | $\langle 1\rangle\Box\Box A \to \Box A$ | | | | (1) |
| 3. | $\langle 1\rangle\neg\Box(\Box A \to A)$ | | | | (1) |
| 4. | $\langle 1,2\rangle\neg(\Box A \to A)$ | | | | (3) |
| 5. | $\langle 1,2\rangle\Box A$ | | | | (4) |
| 6. | $\langle 1,2\rangle\neg A$ | | | | (4) |
| 7. | $\langle 1\rangle\neg\Box\Box A$ | (2) | 8. | $\langle 1\rangle\Box A$ | (2) |
| 9. | $\langle 1,3\rangle\neg\Box A$ | (7) | 11. | $\langle 1,2\rangle A$ | (8) |
| 10. | $\langle 1,3,4\rangle\neg A$ | (9) | | $\otimes$ | |

Not **K**-valid.

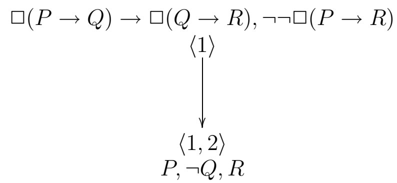





3. a)

| | | | | | |
|---|---|---|---|---|---|
| 1. | $\langle 1\rangle\neg\Big(\big(\Box(P \to Q) \to \Box(Q \to R)\big) \to \neg\Box(P \to R)\Big)$ | | | | |
| 2. | $\langle 1\rangle\Box(P \to Q) \to \Box(Q \to R)$ | | | | (1) |
| 3. | $\langle 1\rangle\neg\neg\Box(P \to R)$ | | | | (1) |
| 4. | $\langle 1\rangle\Box(P \to R)$ | | | | (3) |
| 5. | $\langle 1\rangle\neg\Box(P \to Q)$ | (2) | 6. | $\langle 1\rangle\Box(Q \to R)$ | (2) |
| 7. | $\langle 1,2\rangle\neg(P \to Q)$ | (5) | | | |
| 8. | $\langle 1,2\rangle P$ | (7) | | | |
| 9. | $\langle 1,2\rangle\neg Q$ | (7) | | | |
| 10. | $\langle 1,2\rangle P \to R$ | (4) | | | |
| 11. | $\langle 1,2\rangle\neg P$ (10) \| 12. $\langle 1,2\rangle R$ (10) | | | | |
| | $\otimes$ | | | | |

Not **K**-valid.

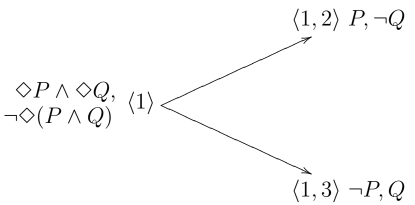


b)

1. $\langle 1\rangle\neg\big((\Diamond P \wedge \Diamond Q) \to \Diamond(P \wedge Q)\big)$
2. $\langle 1\rangle\Diamond P \wedge \Diamond Q$ (1)
3. $\langle 1\rangle\neg\Diamond(P \wedge Q)$ (1)
4. $\langle 1\rangle\Diamond P$ (2)
5. $\langle 1\rangle\Diamond Q$ (2)
6. $\langle 1\rangle\Box\neg(P \wedge Q)$ (3)
7. $\langle 1,2\rangle P$ (4)
8. $\langle 1,2\rangle\neg(P \wedge Q)$ (6)

| | | |
|---|---|---|
| 9. $\langle 1,2\rangle\neg P$ (8) | 10. $\langle 1,2\rangle\neg Q$ (8) | |
| $\otimes$ | 11. $\langle 1,3\rangle Q$ (5) | |
| | 12. $\langle 1,3\rangle\neg(P \wedge Q)$ (6) | |
| | 13. $\langle 1,3\rangle\neg P$ (11) | 13. $\langle 1,3\rangle\neg Q$ (12) |
| | | $\otimes$ |

Not **K**-valid.




c)

1. $\langle 1\rangle\neg\big(\Box(P \wedge Q) \to (\Box P \wedge \Box Q)\big)$
2. $\langle 1\rangle\Box(P \wedge Q)$ (1)
3. $\langle 1\rangle\neg(\Box P \wedge \Box Q)$ (1)

| | |
|---|---|
| 4. $\langle 1\rangle\neg\Box P$ (3) | 5. $\langle 1\rangle\neg\Box Q$ (3) |
| 6. $\langle 1,2\rangle\neg P$ (4) | 10. $\langle 1,2\rangle\neg Q$ (5) |
| 7. $\langle 1,2\rangle P \wedge Q$ (2) | 11. $\langle 1,2\rangle P \wedge Q$ (2) |
| 8. $\langle 1,2\rangle P$ (7) | 12. $\langle 1,2\rangle P$ (11) |
| 9. $\langle 1,2\rangle Q$ (7) | 13. $\langle 1,2\rangle Q$ (11) |
| $\otimes$ | $\otimes$ |



**T-79.5101** **Spring 2008**
**Advanced Course in Computational Logic**
**Exercise Session 7**
**Solutions**

1. a)

1. $\langle 1\rangle\neg\big(\Diamond(P \vee \Diamond\Box\Box P) \to \Diamond P\big)$
2. $\langle 1\rangle\Diamond(P \vee \Diamond\Box\Box P)$ (1)
3. $\langle 1\rangle\neg\Diamond P$ (1)
4. $\langle 1,2\rangle P \vee \Diamond\Box\Box P$ (2)
5. $\langle 1,2\rangle\neg P$ (3)

| | |
|---|---|
| 6. $\langle 1,2\rangle P$ (4) | 7. $\langle 1,2\rangle\Diamond\Box\Box P$ (4) |
| $\otimes$ | 8. $\langle 1,2,3\rangle\Box\Box P$ (7) |
| | 9. $\langle 1,2,3\rangle\Box P$ (8) (reflexivity) |
| | 10. $\langle 1,2,3\rangle P$ (9) (reflexivity) |
| | 11. $\langle 1,2,3\rangle\neg P$ (3) (transitivity) |
| | $\otimes$ |

b)

1. $\langle 1\rangle\neg\Diamond\big(\Diamond P \to \Box(\Diamond P \vee P)\big)$
2. $\langle 1\rangle\neg\big(\Diamond P \to \Box(\Diamond P \vee P)\big)$ (1) (reflexivity)
3. $\langle 1\rangle\Diamond P$ (2)
4. $\langle 1\rangle\neg\Box(\Diamond P \vee P)$ (2)
5. $\langle 1,2\rangle\neg(\Diamond P \vee P)$ (4)
6. $\langle 1,2\rangle\neg\Diamond P$ (5)
7. $\langle 1,2\rangle\neg P$ (5)
8. $\langle 1,2\rangle\neg\big(\Diamond P \to \Box(\Diamond P \vee P)\big)$ (1)
9. $\langle 1,2\rangle\Diamond P$ (8)
10. $\langle 1,2\rangle\neg\Box(\Diamond P \vee P)$ (8)
11. $\langle 1,2,3\rangle P$ (9)
12. $\langle 1,2,3\rangle\neg P$ (6)

$\otimes$

c)

| | | |
|---|---|---|
| 1. | $\langle 1\rangle \neg\Box\big(\Box(\Box P \wedge Q) \to \Diamond\Box\Diamond\Diamond(P \vee Q)\big)$ | |
| 2. | $\langle 1,2\rangle \neg\big(\Box(\Box P \wedge Q) \to \Diamond\Box\Diamond\Diamond(P \vee Q)\big)$ | (1) |
| 3. | $\langle 1,2\rangle \Box(\Box P \wedge Q)$ | (2) |
| 4. | $\langle 1,2\rangle \neg\Diamond\Box\Diamond\Diamond(P \vee Q)$ | (2) |
| 5. | $\langle 1\rangle \Box P \wedge Q$ | (3) (symmetricity) |
| 6. | $\langle 1\rangle \Box P$ | (5) |
| 7. | $\langle 1\rangle Q$ | (5) |
| 8. | $\langle 1\rangle \neg\Box\Diamond\Diamond(P \vee Q)$ | (4) (symmetricity) |
| 9. | $\langle 1,3\rangle \neg\Diamond\Diamond(P \vee Q)$ | (8) |
| 10. | $\langle 1\rangle \neg\Diamond(P \vee Q)$ | (9) (symmetricity) |
| 11. | $\langle 1,3\rangle \neg(P \vee Q)$ | (10) |
| 12. | $\langle 1,3\rangle \neg P$ | (11) |
| 13. | $\langle 1,3\rangle \neg Q$ | (11) |
| 14. | $\langle 1,3\rangle P$ | (6) |
| | $\otimes$ | |

d)

| | | |
|---|---|---|
| 1. | $\langle 1\rangle \neg\Big(\Box P \to \Diamond\big((P \to \Box Q) \to Q\big)\Big)$ | |
| 2. | $\langle 1\rangle \Box P$ | (1) |
| 3. | $\langle 1\rangle \neg\Diamond\big((P \to \Box Q) \to Q\big)$ | (1) |
| 4. | $\langle 1,2\rangle P$ | (2) (serial) |
| 5. | $\langle 1,2\rangle \neg\big((P \to \Box Q) \to Q\big)$ | (3) |
| 6. | $\langle 1,2\rangle P \to \Box Q$ | (5) |
| 7. | $\langle 1,2\rangle \neg Q$ | (5) |

Left branch:

| | | |
|---|---|---|
| 8. | $\langle 1,2\rangle \neg P$ | (6) |
| | $\otimes$ | |

Right branch:

| | | |
|---|---|---|
| 9. | $\langle 1,2\rangle \Box Q$ | (6) |
| 10. | $\langle 1,2,3\rangle Q$ | (9) (serial) |
| 11. | $\langle 1,2,3\rangle \neg\big((P \to \Box Q) \to Q\big)$ | (3) (transitivity) |
| 12. | $\langle 1,2,3\rangle P \to \Box Q$ | (11) |
| 13. | $\langle 1,2,3\rangle \neg Q$ | (11) |
| | $\otimes$ | |



e)

| | | |
|---|---|---|
| 1. | $1\ \neg\Diamond(\Box\Diamond\Box P \to \Box P)$ | |
| 2. | $1\ \neg(\Box\Diamond\Box P \to \Box P)$ | (1) |
| 3. | $1\ \Box\Diamond\Box P$ | (2) |
| 4. | $1\ \neg\Box P$ | (2) |
| 5. | $2\ \neg P$ | (4) |
| 6. | $2\ \Diamond\Box P$ | (3) |
| 7. | $3\ \Box P$ | (6) |
| 8. | $2\ P$ | (7) |
| | $\otimes$ | |

2. Systematic **K**-tableau:

| | | |
|---|---|---|
| 1. | $\langle 1\rangle \neg(\Diamond P \to \Diamond\Box P)$ | |
| 2. | $\langle 1\rangle \Diamond P$ | (1) |
| 3. | $\langle 1\rangle \neg\Diamond\Box P$ | (1) |
| 4. | $\langle 1,2\rangle P$ | (2) |
| 5. | $\langle 1,2\rangle \neg\Box P$ | (3) |
| 6. | $\langle 1\rangle \neg\Diamond\Box P$ | (3) |
| 7. | $\langle 1,2,3\rangle \neg P$ | (5) |

Not **K**-valid.

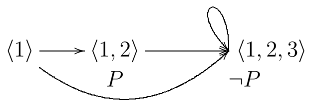
$$\langle 1\rangle \longrightarrow \underset{P}{\langle 1,2\rangle} \longrightarrow \underset{\neg P}{\langle 1,2,3\rangle}$$

Systematic **K4**-tableau:

| | | |
|---|---|---|
| 1. | $\langle 1\rangle \neg(\Diamond P \to \Diamond\Box P)$ | |
| 2. | $\langle 1\rangle \Diamond P$ | (1) |
| 3. | $\langle 1\rangle \neg\Diamond\Box P$ | (1) |
| 4. | $\langle 1,2\rangle P$ | (2) |
| 5. | $\langle 1,2\rangle \neg\Box P$ | (3) |
| 6. | $\langle 1\rangle \neg\Diamond\Box P$ | (3) |
| 7. | $\langle 1,2,3\rangle \neg P$ | (5) |
| 8. | $\langle 1,2,3\rangle \neg\Box P$ | (6) (transitivity) |
| 9. | $\langle 1\rangle \neg\Diamond\Box P$ | (6) |
| 10. | $\langle 1,2,3,4\rangle \neg P$ | (8) |
| 11. | $\langle 1,2,3,4\rangle \neg\Box P$ | (9) (transitivity) |
| 12. | $\langle 1\rangle \neg\Diamond\Box P$ | (9) |
| 13. | $\langle 1,2,3,4,5\rangle \neg P$ | (11) |
| 14. | $\langle 1,2,3,4,5\rangle \neg\Box P$ | (12) (transitivity) |
| 15. | $\langle 1\rangle \neg\Diamond\Box P$ | (12) |
| | $\vdots$ | |

We cannot obtain a complete tableau since an infinite branch is generated into the systematic **K4**-tableau. Since this infinite branch is open, it follows that the formula $\Diamond P \to \Diamond\Box P$ is not **K4**-valid.

Notice that the formulas $\neg P$ and $\neg\Box P$ appear repeatedly in the prefices $\langle 1,2,3\rangle$, $\langle 1,2,3,4\rangle$, and $\langle 1,2,3,4,5\rangle$. Therefore exactly the same formulas hold in the worlds corresponding to these prefices in a countermodel. We attempt to construct a finite countermodel by seeing all such worlds as one. We will then check whether the model that follows is really a countermodel for the claim that the formula given in the exercise is **K4**-valid. When we at the same time assure that the model is based on a transitive frame, we end up with the model




The formula $\Diamond P$ is true in world $\langle 1\rangle$, but $\Diamond\Box P$ is not. Therefore this model is a countermodel for the claim that the formula given in the exercise is **K4**-valid.

3.

| | | | | | |
|---|---|---|---|---|---|
| 1. | $\langle 1\rangle\neg\big((\Diamond P \to \Diamond\Diamond P) \wedge \neg P\big)$ | | | | |
| 2. | $\langle 1\rangle\neg(\Diamond P \to \Diamond\Diamond P)$ | (1) | 3. | $\langle 1\rangle\neg\neg P$ | (1) |
| 6. | $\langle 1\rangle\Diamond P$ | (2) | 4. | $\langle 1\rangle P$ | (3) |
| 7. | $\langle 1\rangle\neg\Diamond\Diamond P$ | (2) | 5. | $\langle 1\rangle\neg P$ | (LP) |
| 8. | $\langle 1,2\rangle P$ | (6) | | $\otimes$ | |
| 9. | $\langle 1,2\rangle\neg\Diamond P$ | (7) | | | |
| 10. | $\langle 1,2\rangle\Box\neg P \to \neg P$ | (GP) | | | |
| 11. | $\langle 1,2\rangle\neg\Box\neg P$ | (10) | 12. | $\langle 1,2\rangle\neg P$ | (10) |
| 13. | $\langle 1,2,3\rangle\neg\neg P$ | (11) | | $\otimes$ | |
| 14. | $\langle 1,2,3\rangle P$ | (12) | | | |
| 15. | $\langle 1,2,3\rangle\neg P$ | (9) | | | |
| | $\otimes$ | | | | |