T-79.5201 Discrete Structures, Autumn 2006

Home assignment 2 (due 29 Nov at 12:15 p.m.)

1. Determine algebraic expressions for the following generating functions, based directly on the structure of the respective combinatorial families:

   (a) The egf for sequence $\langle a_n \rangle$, where $a_n =$ the number of ways an n-element ground set can be partitioned into (unordered) pairs of exactly two elements. (In other words, $a_n =$ the number of *perfect matchings* of a complete n-node graph.)

   (b) The egf for sequence $\langle b_n \rangle$, where $b_n =$ the number of "binary tree partitions" of the set $[n] = \{1, \ldots, n\}$, i.e. the number of labelled binary trees where each node contains some nonempty subset of the set $[n]$, these subsets are disjoint and together cover all of $[n]$. (A *binary tree* is an ordered rooted tree, where each nodes has two descendant subtrees, either or both of which may be empty. By direct drawing and counting one observes that $b_0 = 1$, $b_1 = 1$, $b_2 = 5$, $b_3 = 43$ etc.)

2. Show that if a combinatorial family $\mathcal{B}$ can be decomposed as $\mathcal{B} \xrightarrow{\sim} \mathcal{A}^{[*]}$, then the counts of objects in families $\mathcal{B}$ and $\mathcal{A}$ of different weights are related by:

   $$nb_n = \sum_{k=0}^{n} \binom{n}{k} k a_k b_{n-k}.$$

   (*Hint:* The "$zD \log$" trick.) Based on this result, derive a recurrence formula for the number c_n of connected labelled graphs with n nodes, and use your formula to compute the values $c_1, \ldots, c_6$. (*Hint:* Determine first the *total* number g_n of all n-node graphs.)

3. Determine the number of strings of length n generated by the context free grammar

   $$S \to aSS \mid bS \mid cS \mid d$$

   (If you are not familiar with the grammar formalism, please consult the course personnel.)

4. Estimate the value of the sum $\sum_{k=1}^{n} k \ln k$ up to order $O(1)$. (*Hint:* Consider first the sum with an upper bound of $n - 1$ instead of n.) What estimate can you derive from this for the rate of growth of the product $1^1 \cdot 2^2 \cdots n^n$ as a function of n?