T-79.5201 Discrete Structures, Autumn 2007
Home assignment 1 (due 7 Nov at 12:15 p.m.)

1. Consider the family of all pairs (A, B) of disjoint k-element subsets of $[n] = \{1, \ldots, n\}$. A set $Y \subseteq [n]$ *separates* the pair (A, B) if $A \subseteq Y$ and $B \cap Y = \emptyset$. Show that there exists a family of $\ell = 3k4^k \ln n$ sets such that every pair (A, B) is separated by at least one of them. (*Hint:* Consider a uniform random family of ℓ subsets of $[n]$. Estimate the probability that none of them separates a given pair (A, B).)

2. Let F be a Boolean formula in conjunctive normal form, with n variables and m clauses.[1]

   (a) Let k be the minimum number of literals in any of the clauses in F. Show that there is a truth assignment to the variables of F that satisfies at least $(1 - 2^{-k})m$ of the clauses. (*Hint:* Linearity of expectation.)

   (b) Formula F is *2-satisfiable* if any two of its clauses can be simultaneously satisfied. Show that in this case there is a truth assignment to the variables that satisfies at least γm of the clauses, where $\gamma = (\sqrt{5} - 1)/2$. (*Hint:* Consider a random truth assignment to the variables, biased so that if a literal $x^{\pm}$ appears as a unary clause in F then $\Pr(x^{\pm} = 1) = \gamma$, otherwise $\Pr(x^{\pm} = 1) = 1/2$.)

3. Consider the space Ω_n of random equiprobable permutations of $[n] = \{1, \ldots, n\}$. A permutation $\pi \in \Omega_n$ contains an *increasing subsequence of length* k, if there are indices $i_1 < \cdots < i_k$ such that $\pi(i_1) < \cdots < \pi(i_k)$.

   (a) Show that a.a.s. a random permutation $\pi \in \Omega_n$ does not contain an increasing subsequence of length $\geq e\sqrt{n}$. (*Hint:* First-moment method.)

   (b) Denote the length of a maximal increasing subsequence contained in a permutation π by $I(\pi)$, and correspondingly the length of a maximal *decreasing* subsequence by $D(\pi)$. Erdős and Szekeres proved in 1935 that $I(\pi)D(\pi) \geq n$ for any permutation π of $[n]$.[2] Deduce from this result and the result of part (a) that a.a.s. a random permutation $\pi \in \Omega_n$ contains an increasing subsequence of length $\geq \sqrt{n}/e$.

4. [Zarankiewicz's Problem.] Let $k_a(n)$ be the minimal k such that all $n \times n$ 0-1 matrices containing more than k ones contain an $a \times a$ submatrix consisting entirely of ones (an "all-ones" submatrix). It is known that for all n and a,

$$k_a(n) \leq (a-1)^{1/a} n^{2-1/a} + (a-1)n.$$

   Establish a corresponding lower bound: for every constant $a \geq 2$ there is an $\epsilon > 0$ such that $k_a(n) \geq \epsilon n^{2-2/a}$. (*Hint:* Alteration method. Take a random $n \times n$ 0-1 matrix A, where each entry has probability $p = n^{-2/a}$ of being 1. Associate with each $a \times a$ submatrix e of A an indicator variable $Y_e \sim$ "e is an all-ones submatrix". Kill all all-ones submatrices by switching one entry in each to 0.)

---

[1] If you are not familiar with these notions, please ask the lecturer and/or your colleagues.

[2] You do not need to prove this claim, but in fact it has a very simple and elegant proof; think about it or look it up in any combinatorics textbook under "Ramsey theory."