T-79.5201 Discrete Structures, Autumn 2007
Home assignment 3 (due 19 Dec at 4:00 p.m.)

1. In the *bin-packing problem*, we are given a set of items with sizes $a_1, \ldots, a_n$, with $0 \leq a_i \leq 1$ for $i = 1, \ldots, n$. The goal is to pack the items in a minimum number of bins, each with a total capacity of 1 unit. Let each of the a_i be chosen independently at random according to some distribution, which might be different for different i. Let X be the minimum number of bins required in the best packing of the resulting set of items, and $\mu = E[X]$ the expected number of bins in the best packing. For a given $\lambda > 0$, derive a bound on the probability $\Pr(|X - \mu| > \lambda)$.

2. [Baranyai's Rounding Lemma.] Let $A = (a_{ij})$ be a matrix of real elements. Prove that there exists an integer matrix $\hat{A} = (\hat{a}_{ij})$ satisfying $|a_{ij} - \hat{a}_{ij}| < 1$ for all i, j, such that:

   $|\sum_j a_{ij} - \sum_j \hat{a}_{ij}| < 5$ for all i,
   $|\sum_i a_{ij} - \sum_i \hat{a}_{ij}| < 5$ for all j, and
   $|\sum_i \sum_j a_{ij} - \sum_i \sum_j \hat{a}_{ij}| < 5$.

   (In Baranyai's 1974 result, the constant 5 is replaced by 1.)

3. By the result of Problem 2(a) in Home Assignment 1, for any k-cnf formula F there is a truth assignment to the variables of F that satisfies at least a fraction $(1 - 2^{-k})$ of its clauses. Design an efficient algorithm that, given such a formula, actually *finds* the promised assignment.

4. [Alon & Spencer, Prob. 15.3:] Prove that if $\mathcal{F} = \{S_1, \ldots, S_m\}$ is a hypergraph on n vertices satisfying $\sum_{i=1}^{m} 2^{1-|S_i|} < 1$, then $\mathcal{F}$ is two-colourable. In the case $m = n$ design an efficient algorithm that, given $\mathcal{F}$, determines an appropriate two-colouring of the vertices that leaves no $S_i \in \mathcal{F}$ monochromatic.