6. Locally Dependent Events

- The "Lovász Local Lemma" (Erdős & Lovász 1975).
- Consider a probabilistic proof for the existence of a "good" configuration (sample point) $\omega \in \Omega$ that has none of the "bad" properties $A_1, \dots, A_n$; i.e. none of the events $A_1, \dots, A_n$ occur in ω, i.e. $\omega \in \bigcap_i \bar{A}_i$.
- The aim is thus to prove that $\Pr(\bigcap_i \bar{A}_i) > 0$.
- If each of the bad events has prob. $\Pr(A_i) = p_i < 1$, <u>and the events are mutually independent</u>, then

$$\Pr(\bigcap_i \bar{A}_i) = \prod_i (1-p_i) > 0,$$

and we are done.

- However, if the events A_i have dependencies, we can only argue that

$$\Pr(\bigcap_i \bar{A}_i) = \Pr(\overline{\bigcup_i A_i}) = 1 - \Pr(\bigcup_i A_i) \geq 1 - \sum_i \Pr(A_i).$$

Also this is OK, if there are not too many events A_i, and/or the probabilities $p_i = \Pr(A_i)$ are small. (And in the general case this is the best possible bound, viz. when the events are disjoint, $A_i \cap A_j = \emptyset$ for $i \neq j$.)

- Nevertheless one can get better bounds when the dependencies among events are <u>local</u>. (E.g. in random graphs, events pertaining to disjoint vertex sets are [often] independent, and a given vertex set intersects only a bounded number of others.)

- A dependency graph for a set of events $A_1, \dots, A_n$ is a graph G on vertex set $[n] = \{1, \dots, n\}$ s.th. event A_i is mutually independent of all the events A_j s.th. $j \not\sim i$ in G (and also all their Boolean combinations).

- Theorem 6.1 (LLL, Erdős & Lovász 1975). Let G be a dependency graph for events $A_1, \dots, A_n$. Assume that $\deg(i) \le d$ and $\Pr(A_i) \le p$ for all $i = 1, \dots, n$. If $4pd \le 1$, then $\Pr(\bigcap_i \bar{A}_i) > 0$.

Proof. We first prove by induction on m that for any m events $A_1, \dots, A_m$ (re-indexed as necessary):

$$\Pr(A_1 \mid \bar{A}_2 \cdots \bar{A}_m) \le 2p.$$

| $\bar{A}_2 \cdots \bar{A}_m$ shorthand for $\bar{A}_2 \cap \dots \cap \bar{A}_m$

For $m = 1$, $\Pr(A_1) \le p \le 2p$ by assumption.

For $m > 1$, let for a given event A_1, events $A_2, \dots, A_k$ be all its neighbours according to G. (Note $k-1 \le d$.) Then, applying the decomposition $\Pr(A \mid BC) = \Pr(AB \mid C)/\Pr(B \mid C)$:

$$(*) \quad \Pr(A_1 \mid \bar{A}_2 \cdots \bar{A}_m) = \frac{\Pr(A_1 \bar{A}_2 \cdots \bar{A}_k \mid \bar{A}_{k+1} \cdots \bar{A}_m)}{\Pr(\bar{A}_2 \cdots \bar{A}_k \mid \bar{A}_{k+1} \cdots \bar{A}_m)}.$$

The numerator of $(*)$ can be upperbounded by independence:

$$\Pr(A_1 \bar{A}_2 \cdots \bar{A}_k \mid \bar{A}_{k+1} \cdots \bar{A}_m) \le \Pr(A_1 \mid \bar{A}_{k+1} \cdots \bar{A}_m)$$
$$= \Pr(A_1) \le p$$

and denominator lowerbounded by induction:

$$\Pr(\bar{A}_2 \cdots \bar{A}_k \mid \bar{A}_{k+1} \cdots \bar{A}_m) = 1 - \Pr\Big(\bigcup_{i=2}^{k} A_i \;\Big|\; \bar{A}_{k+1} \cdots \bar{A}_m\Big)$$
$$\ge 1 - \sum_{i=2}^{k} \Pr(A_i \mid \bar{A}_{k+1} \cdots \bar{A}_m)$$
$$\ge 1 - (k-1) \cdot 2p \ge \tfrac{1}{2}.$$

| Note: $k-1 \le d$, $2pd \le 1/2$

Thus,

$$\Pr(A_1 \mid \bar{A}_2 \cdots \bar{A}_m) \le \frac{p}{1/2} = 2p,$$

completing the induction.

To obtain the claim of the theorem, note that

$$\begin{aligned}\Pr(\bar{A}_1 \cdots \bar{A}_n) &= \prod_{i=1}^{n} \Pr(\bar{A}_i \mid \bar{A}_1 \cdots \bar{A}_{i-1}) \\ &\geq (1-2p)^n \\ &> 0. \quad \square\end{aligned}$$

- <u>Theorem</u> 6.2 (Generalised LLL, Spencer 1977). Let G be a dependency graph for events $A_1, \ldots, A_n$. Assume that there exist real numbers $x_1, \ldots, x_n$, $0 \le x_i < 1$, s.th. for all i:

$$\Pr(A_i) \le x_i \cdot \prod_{j \sim i} (1-x_j).$$

Then

$$\Pr\Big(\bigcap_i \bar{A}_i\Big) \geq \prod_{i=1}^{n} (1-x_i).$$

<u>Proof.</u> As above, with the induction hypothesis

$$\Pr(A_1 \mid \bar{A}_2 \cdots \bar{A}_m) \le x_1,$$

and lowerbounding the denominator in (*) as

$$\begin{aligned}\Pr(\bar{A}_2 \cdots \bar{A}_k \mid \bar{A}_{k+1} \cdots \bar{A}_m) &= \prod_{j=2}^{k} \Pr(\bar{A}_j \mid \bar{A}_{j+1} \cdots \bar{A}_m) \\ &\geq \prod_{j=2}^{k} (1-x_j) \\ &= \prod_{j \sim 1} (1-x_j). \quad \square\end{aligned}$$

• Example 1. Ramsey numbers.

Recall: $R(k) > n$ if there is a two-colouring of the edges of K_n that induces no monochromatic k-cliques.

Consider a random two-colouring of K_n, and associate to each k-set of vertices $S \subseteq [n]$ the event $A_S \sim$ "S is monochromatic". Define a dependency graph for these events by

$$S \sim T \text{ if } |S \cap T| \geq 2.$$

Applying the LLL with $\Pr(A_S) = 2 \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$ and

$$d = \binom{k}{2}\binom{n}{k-2}$$

yields:

Theorem 6.3 If $4 \cdot 2^{1-\binom{k}{2}} \cdot \binom{k}{2}\binom{n}{k-2} < 1$, then $R(k) > n$. □

Working out the asymptotics, this results in:

$$R(k) > \frac{\sqrt{2}}{e} \cdot k 2^{k/2} (1 + o(1)).$$

This is a factor of $\sqrt{2}$ improvement over the bound obtained by the alteration method (Thm 3.1)

- <u>Example</u> 2. Hypergraph 2-colourings.

A <u>hypergraph</u> with n vertices is any family $\mathcal{F}$ of subsets of $[n] = \{1, \dots, n\}$.

A hypergraph $\mathcal{F}$ is k-<u>uniform</u> if every "hyperedge" $S \in \mathcal{F}$ has exactly k elements. (Thus, a standard graph is a 2-uniform hypergraph.)

A hypergraph $\mathcal{F}$ is r-<u>regular</u> if every vertex is contained in exactly r hyperedges.

A hypergraph $\mathcal{F}$ is 2-<u>colourable</u> (or "has property B", where B stands either for [Felix] Bernstein or "bipartite"), if the vertices can be 2-coloured so that no $S \in \mathcal{F}$ is monochromatic.

- <u>Theorem</u> 6.4 (Erdös 1963). Every k-uniform hypergraph with fewer than 2^{k-1} edges is 2-colourable.

<u>Proof.</u> Let $\mathcal{F}$ be a k-uniform hypergraph with $|\mathcal{F}| < 2^{k-1}$. Consider a random 2-colouring of the vertices of $\mathcal{F}$. Then for each $S \in \mathcal{F}$, $\Pr(S \text{ is mono-}\chi) = 2 \cdot 2^{-k}$, and so:

$$\Pr(\exists S \in \mathcal{F} : S \text{ mono-}\chi) \le \sum_{S \in \mathcal{F}} \Pr(S \text{ mono-}\chi)$$

$$< 2^{k-1} \cdot 2^{1-k} = 1. \quad \square$$

Note how this proof breaks down if $|\mathcal{F}| \geq 2^{k-1}$, and hence especially if $|\mathcal{F}|$ increases with the number of vertices n.

Now if the edges $S \in \mathcal{F}$ were disjoint (nonintersecting), then the events $A_S \sim$ "S mono-χ" would be mutually independent, and one would have

$$\Pr(\forall S \in \mathcal{F} : S \text{ not mono-}\chi) = \prod_{S \in \mathcal{F}} \Pr(S \text{ not mono-}\chi) = (1 - 2^{1-k})^{|\mathcal{F}|} > 0.$$

The case of nonintersecting edges is of course not very interesting, but with the LLL one can easily prove the following nontrivial result:

Theorem 6.5 (Erdös & Lovász 1975). Let $\mathcal{F}$ be a k-uniform hypergraph where every edge intersects at most 2^{k-3} other edges. Then $\mathcal{F}$ is 2-colourable.

Proof Consider a random 2-colouring of the vertices of $\mathcal{F}$ and events $A_S \sim$ "S is mono-χ" for each $S \in \mathcal{F}$. Define a dependency graph for the A_S by

$$S \sim T \text{ if } S \cap T \neq \emptyset.$$

Apply the LLL with $p = \Pr(A_S) = 2^{1-k}$, $d = 2^{k-3}$. Then $4pd = 4 \cdot 2^{1-k} \cdot 2^{k-3} = 1$, and we immediately obtain:

$$\Pr(\forall S \in \mathcal{F} : S \text{ not mono-}\chi) = \Pr\Big(\bigcap_{S \in \mathcal{F}} \bar{A}_S\Big) > 0.$$

$\square$