Helsinki University of Technology
Laboratory for Theoretical Computer Science
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T-79.5204 Combinatorial Models and Stochastic Algorithms (6 cr)
Exam Thu 10 May 2007, 1–4 p.m.

Permitted material at exam: lecture notes, any personal handwritten notes, tutorial problems and their solutions; calculator.

Write down on each answer sheet:
- Your name, department, and study book number
- The text: "T-79.5204 Combinatorial Models and Stochastic Algorithms 10.5.2007"
- The total number of answer sheets you are submitting for grading

1. Prove that the graph property "G contains a $k \times k$ torus" (i.e. a subgraph isomorphic to a $k \times k$ lattice with periodic boundary conditions) has a threshold function for any fixed $k \geq 2$, and compute it. *7p.*

2. Consider the space $S = \{0,1\}^n$ of binary strings of length n, and denote by $k(x)$ the *weight*, i.e. the number of ones in a string $x \in S$. Design some Markov chain Monte Carlo sampling method that (asymptotically) samples strings in S in proportion to their weight, i.e. the stationary distribution of the sampler satisfies $\Pr(x) = c \cdot k(x)$, where c is the appropriate normalisation constant. Describe clearly what are the transition probabilities for your sampling chain, and justify its regularity. *8p.*

3. Consider a lone rook (Finnish "torni") making random moves on an $n \times n$ chessboard, meaning that at each move, the rook chooses one of its permissible next-state squares uniformly at random. Show that for $n \geq 3$ the Markov chain defined by these moves is regular, and determine its stationary distribution. Calculate some upper bound on the mixing time of the chain. *8p.*

4. Consider the following *Exact k-Hitting Set* problem: Given a family $\{C_1, \ldots, C_m\}$ of k-element subsets of a finite set S, $|S| = n$, is there a subset $H \subseteq S$ such that each C_i, $i = 1, \ldots, m$, contains *exactly one* element of H (i.e., H "hits" each one of the sets C_i exactly once). The problem is NP-complete for $k \geq 3$. Make an educated guess concerning the location of "hard instances" for this problem. (NB. In the case $k = 2$ the problem is equivalent to asking whether a given graph is bipartite, which can easily be determined in polynomial time.) *7p.*

Total 30p.