T-79.5204 **Spring 2007**
Combinatorial Models and Stochastic Algorithms
Tutorial 8, March 22
Problems

1. Let Q be a monotone graph property and denote by $P_n^Q(p)$ the probability that a random graph $G \in \mathcal{G}(n,p)$ has property Q. Prove that $P_n^Q(p)$ is a continuous, increasing function of p. (*Hint:* The result may not be as obvious as it seems. One approach is to write down an explicit expression for the probability that a subgraph with exactly k edges has property Q, sum over all k, and differentiate with respect to p. Other, more purely combinatorial approaches are also possible.)

2. Prove that the graph property "G has maximum degree at least k" has a threshold function for $k \geq 1$, and compute it.

3. Prove that the graph property "G contains a d-dimensional cube" has a threshold function for $d \geq 1$, and compute it. (The "d-dimensional cube" has 2^d vertices represented as $\{0,1\}^d$, and two vertices are connected by an edge if and only if their representations differ in exactly one position.)

4. Let X be the number of cycles in a random graph $G \in \mathcal{G}(n,p)$, where $p = c/n$. Find an exact formula for $E[X]$, and estimate the asymptotics of $E[X]$ when (a) $c < 1$ and (b) $c = 1$.

5. Consider the space Ω_n of random equiprobable permutations of $[n] = \{1,\ldots,n\}$. A permutation $\pi \in \Omega_n$ contains an *increasing subsequence of length* k, if there are indices $i_1 < \cdots < i_k$ such that $\pi(i_1) < \cdots < \pi(i_k)$.

   (a) Show that almost no permutation $\pi \in \Omega_n$ contains an increasing subsequence of length at least $e\sqrt{n}$. (*Hint:* Let $I_k(\pi)$ be the number of increasing subsequences of length k contained in π. Estimate $E[I_k]$.)

   (b) Denote the length of a maximal increasing subsequence contained in a permutation π by $I(\pi)$, and correspondingly the length of a maximal *decreasing* subsequence by $D(\pi)$. Erdős and Szekeres proved in 1935 that $I(\pi)D(\pi) \geq n$ for any permutation π of $[n]$.[1] Deduce from this result and the result of part (a) that almost every permutation $\pi \in \Omega_n$ contains an increasing subsequence of length at least $\sqrt{n}/e$.

[1] You do not need to prove this claim, but in fact it has a very simple and elegant proof; think about it or look it up in any combinatorics textbook under "Ramsey theory."