T-79.5204 **Spring 2007**
Combinatorial Models and Stochastic Algorithms
Tutorial 10, April 12
Problems

1. Verify by direct calculation that in the simulated annealing algorithm the finite-temperature Gibbsian distributions $\pi^{(T)}$ for $T > 0$ do indeed converge pointwise to the desired limit distribution π^* as $T \to 0$.

2. Consider a simple state space graph with states $S = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ and neighbourhood structure $N(\sigma_i) = \{\sigma_{i-1}, \sigma_{i+1}\}$, where the indices are computed modulo 4. Write down explicitly the transition probability matrix of the simulated annealing algorithm at temperature t for this system, when the function to be minimised is given by $H(\sigma_0) = 1$, $H(\sigma_1) = 2$, $H(\sigma_2) = 0$, $H(\sigma_3) = 2$. Given a cooling schedule where the temperature at step k of the algorithm is $t_k > 0$, what is the probability that the algorithm when initialised in the locally optimal state σ_0 will stay there forever? Find a sequence t_k for which this probability is nonzero. What kind of cooling schedule would, according to Theorem 8.5 (p. 96 of the notes) guarantee asymptotic convergence to the globally optimal state σ_2?

3. The NP-complete PARTITION problem is defined as follows: given a sequence of $2n$ nonnegative integers $x_1, \ldots, x_{2n}$, is there a subsequence of n numbers whose sum is exactly half the sum of the whole sequence? Formulate the task of finding approximate partitions of an integer sequence as a minimisation problem, and present a simulated annealing approach to solving it. What kinds of cooling schedules would Theorem 8.5 suggest for your algorithm in the case of input sequences consisting of $2n$ numbers from the interval $[0, N]$? (You might consider also actually implementing your algorithm and experimenting with some more realistic cooling schedules.)

4. Consider a simple self-reduction setting for an NP relation R, where for any input x of length $|x| > n_0$, the set of witnesses $R(x) = \{w \mid R(x, w)\}$ can be partitioned into two disjoint classes by polynomially computable length-decreasing self-reduction functions f_0 and f_1, i.e. for $|x| > n_0$,

$$R(x) = R(f_0(x)) \uplus R(f_1(x)), \quad |f_0(x)|, |f_1(x)| < |x|.$$

Assume the availability of a perfect small-scale sampler $U_R(x)$ for generating elements $w \in R(x)$ uniformly at random for inputs x of length $|x| \leq n_0$, and an FPRAS $A(x, \epsilon)$ for approximately counting the number of elements in $R(x)$ for all x. Show how these can be combined to obtain an FPAUS $S(x, \delta)$ for sampling elements in $R(x)$ almost uniformly at random for arbitrary inputs x. (For simplicity, you may assume that $A(x, \epsilon)$ provides its answers with perfect reliability, rather than reliability $\frac{3}{4}$ as would be permitted by the general FPRAS definition.)

5. Continuing the previous problem setting, assume conversely the availability of a perfect small-scale witness-counter $N_R(x)$ for computing the size of $R(x)$ for $|x| \leq n_0$, and an FPAUS $S(x, \delta)$ for sampling elements in $R(x)$ almost uniformly at random for all x. Show how these can be combined to obtain an FPRAS $A(x, \epsilon)$ for approximately counting the number of elements in $R(x)$ for arbitrary inputs x.