A sufficient condition for this to hold is that $k \geq k(n,\varepsilon) = (2+\varepsilon)\frac{\ln n}{\ln 1/q}$. Thus for large n, almost no graph $G \in \mathcal{G}(n,p)$ can have a colouring that would assign the same colour to $k(n,\varepsilon)$ or more nodes. Hence, a proper colouring of almost any $G \in \mathcal{G}(n,p)$ requires at least $\frac{n}{k(n,\varepsilon)} = \frac{\ln 1/q}{2+\varepsilon} \cdot \frac{n}{\ln n}$ colours. □

Theorem 7.7 *Let p, $0 < p < 1$ be constant. Then for a.e. $G \in \mathcal{G}(n,p)$:*

$$\omega(G) \in \{d, d+1\},$$

where $d = d(n,p)$ is the largest integer such that

$$\binom{n}{d} p^{\binom{d}{2}} \geq \ln\ n.$$

(This implies $d = 2\log_{1/p}(n) + O(\log\log n.)$.) □

A *graph property* Q is an isomorphism-closed family of graphs, i.e. if $G \in Q$ (or "G has Q") and $G \approx G'$, then also $G' \in Q$.

A *threshold function* for a graph property Q is a function $t : \mathbb{N} \to \mathbb{R}$ such that

$$\Pr(G \in \mathcal{G}(n,p(n)) \text{ has } Q) \xrightarrow[n\to\infty]{} \begin{cases} 1, & \text{if } p \succ t, \\ 0, & \text{if } p \prec t, \end{cases}$$

where:

$$p \succ t \Leftrightarrow \lim_{n\to\infty} \frac{p(n)}{t(n)} = \infty,$$
$$p \prec t \Leftrightarrow \lim_{n\to\infty} \frac{p(n)}{t(n)} = 0.$$

Further notation:

$$p \sim t \Leftrightarrow \lim_{n\to\infty} \frac{p(n)}{t(n)} = 1,$$
$$p \approx t \Leftrightarrow p(n) = \Theta(t(n)).$$

Denote: $P_n^Q(p) = \Pr(G \in \mathcal{G}(n,p) \text{ has } Q)$.

For technical reasons, we will actually use the following slightly stronger definition for a threshold function: $t(n)$ is a threshold function for graph property Q if

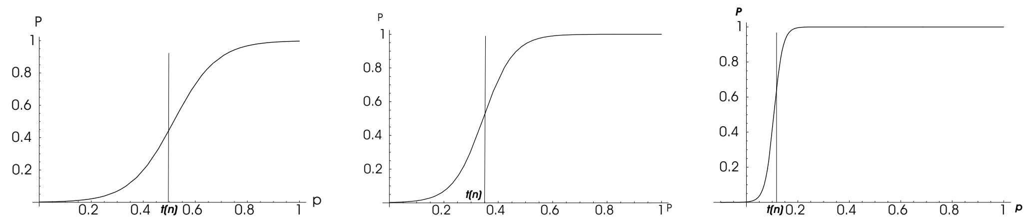

Figure 7: $P_n^Q(p)$ for (a) small, (b) intermediate and (c) large n.

for any sequence $n_1 < n_2 < \dots$ of graph sizes and $p(n_1), p(n_2), \dots$ of associated edge probabilities,

$$\lim_{k\to\infty} \frac{p(n_k)}{t(n_k)} = \infty \Rightarrow P_{n_k}^Q(p(n_k)) = 1, \qquad (*)$$

$$\lim_{k\to\infty} \frac{p(n_k)}{t(n_k)} = 0 \Rightarrow P_{n_k}^Q(p(n_k)) = 0. \qquad (**)$$

A graph property is *monotone* if it is preserved under addition of edges, i.e. if $G = (V,E)$ and $G' = (V,E')$ are graphs such that $E \subseteq E'$ and G has Q, then also G' has Q. For monotone Q it is the case that $p_1 \le p_2 \Rightarrow P_n^Q(p_1) \le P_n^Q(p_2)$, so the inverse of $P_n^Q(p)$ is well-defined:

$p_n^Q(\alpha) =$ the smallest p such that $P_n^Q(p) \ge \alpha$.

In fact for monotone Q one can show that $P_n^Q(p)$ is a continuous, strictly increasing function of p, so actually $p_n^Q(\alpha) =$ *unique* p such that $P_n^Q(p) = \alpha$.

Figure 7 illustrates the evolution of the function P_n^Q, and a corresponding threshold function $t(n)$, for a monotone graph property Q from small to large values of n.

Lemma 7.8 *A function $t(n)$ is a threshold for monotone graph property Q if and only if $t(n) \approx p_n^Q(\alpha)$ for all $0 < \alpha < 1$.*

Proof. Suppose that $t(n)$ is threshold function for Q, but $t(n) \not\approx p_n^Q(\alpha)$ for some $0 < \alpha < 1$. Denoting for brevity $p(n) = p_n^Q(\alpha)$, this means that either there is a sequence $n_1, n_2, \dots$ such that

$$p(n_k)/t(n_k) \to \infty,$$

or there is a sequence $n_1, n_2, \dots$ such that

$$p(n_k)/t(n_k) \to 0.$$

However, since for all n it holds that $P_n^Q(p(n)) = P_n^Q(p_n^Q(\alpha)) = \alpha$, $0 < \alpha < 1$, the former case violates condition (*) and the latter case condition (**) in the definition of a threshold function.

"$\Leftarrow$" Assume then that $t(n)$ is *not* a threshold function for Q. Then there are either a sequence $n_1, n_2, \ldots$ and a constant $\alpha < 1$ such that

$$p(n_k)/t(n_k) \to \infty \quad \text{but} \quad P_{n_k}^Q(p(n_k)) \leq \alpha,$$

or a sequence $n_1, n_2, \ldots$ and a constant $\alpha > 0$ such that

$$p(n_k)/t(n_k) \to 0 \quad \text{but} \quad P_{n_k}^Q(p(n_k)) \geq \alpha.$$

In the former case,

$$t(n_k) \prec p(n_k) \leq p_{n_k}^Q(\alpha),$$

and in the latter case

$$t(n_k) \succ p(n_k) \geq p_{n_k}^Q(\alpha).$$

Thus in either case, $t(n) \not\approx p_n^q(\alpha)$ for some $0 < \alpha < 1$. □

Theorem 7.9 *Every monotone graph property Q has a threshold function.*

Proof. For brevity, denote $p_n^Q(\alpha) = p(\alpha)$. Choose some arbitrary $0 < \alpha < \frac{1}{2}$. The goal is to prove that $p(\alpha) \approx p(1-\alpha)$, thus establishing e.g.

$$t(n) = p\left(\frac{1}{2}\right) = p_n^Q\left(\frac{1}{2}\right)$$

as a threshold function for Q. (Since $p(\alpha) \leq p(\frac{1}{2}) \leq p(1-\alpha)$.)

Let $m \in \mathbb{N}$ be such that $(1-\alpha)^m \leq \alpha$. Let $p = p_n(\alpha)$ and consider a sample of m independent graphs $G_1, \ldots, G_m$ from $\mathcal{G}(n, p)$. Then the graph $G_1 \cup \cdots \cup G_m \in \mathcal{G}(n, q)$, where $q = 1 - (1-p)^m \leq mp$, and so

$$\Pr(G_1 \cup \cdots \cup G_m \text{ has } Q) \leq \Pr(G \in \mathcal{G}(n, mp_n(\alpha)) \text{ has } Q).$$

On the other hand, since Q is monotone, if any G_i has Q, then so does $G_1 \cup \cdots \cup G_m$. Thus,

$$\begin{aligned}\Pr(G_1 \cup \cdots \cup G_m \text{ does not have } Q) &\leq (1 - \Pr(G_i \text{ has } Q))^m \\ &= (1-\alpha)^m \leq \alpha.\end{aligned}$$

Hence,

$$\Pr_n^Q(mp_n(\alpha)) \geq \Pr(G_1 \cup \cdots \cup G_m \text{ has } Q) \geq 1-\alpha,$$

and so

$$p_n(\alpha) \leq p_n(1-\alpha) \leq mp_n(\alpha),$$

i.e. $p(\alpha) \approx p(1-\alpha)$. (Since m depends only on α, not on n.) $\square$

Consider a graph property Q defined as "G has Q" if $X(G) > 0$, where $X \geq 0$ is a random variable on $\mathcal{G}(n,p)$.

E.g. if $X(G)$ denotes the number of spanning trees of G, then property Q corresponds to connectedness.

Recall the two properties characterising a threshold function $t(n)$:

(i) $p(n) \prec t(n) \Rightarrow$ almost no $G \in \mathcal{G}(n,p(n))$ has Q.
(ii) $p(n) \succ t(n) \Rightarrow$ almost all $G \in \mathcal{G}(n,p(n))$ have Q.

If X is integral, then one can aim to verify conditions (i) and (ii) by the so called "first-moment method" and "second-moment method", respectively.

The first-moment method consists simply of upper-bounding the expectation $E[X]$ and applying Markov's inequality:

$$\Pr(X \geq 1) \leq E[X] \qquad (\text{ more generally, for } a > 0 \;\; p(X \geq a) \leq E[X]/a\ ).$$

More specifically, one aims to show that if the choice of edge probabilities satisfies $p(n) \prec t(n)$, then $E[X_n] \to 0$. By Markov's inequality it then follows that also $P_n^Q(p(n)) = \Pr(X_n \geq 1) \to 0$.

The second-moment method is based on lower-bounding $E[X]$ *and* upper-bounding $\mathrm{Var}[X]$.

Denote $\mu_n = E[X_n]$, $\sigma_n^2 = \mathrm{Var}[X_n] = E[(X_n-\mu_n)^2] = E[X_n^2] - \mu_n^2$. Recall Chebyshev's inequality (a simple consequence of Markov's inequality): for any $\lambda > 0$,

$$\Pr(|X-\mu| \geq \lambda) \leq \frac{\sigma^2}{\lambda^2}.$$

Lemma 7.10 *If $\mu_n > 0$ for n large, and $\frac{\sigma_n^2}{\mu_n^2} \to 0$ as $n \to \infty$, then $\Pr(X_n > 0) \to 1$ as $n \to \infty$.*

Proof. If $X_n = 0$, then $|X_n - \mu_n| = \mu_n$. Hence

$$\Pr(X_n = 0) \leq \Pr(|X_n - \mu_n| \geq \mu_n) \leq \frac{\sigma_n^2}{\mu_n^2} \to 0 \text{ as } n \to \infty. \quad \square$$

For the next result, denote the number of nodes in a graph G by $|G|$, the number of edges by $e(G)$, and define its *density* as $\delta(G) = \frac{e(G)}{|G|}$. Aay that a graph G is *balanced* if $\delta(G') \leq \delta(G)$ for all subgraphs G' of G.

Theorem 7.11 *Let H be a balanced graph. Then the graph property "G has a subgraph isomorphic to H" has threshold function $n^{-1/\delta(H)}$.*

Proof. Denote $X(G) =$number of H-subgraphs of a given graph G. Let $k = |H|$, $l = e(H)$, so $\delta(H) = l/k$, and let $G \in \mathcal{G}(n, p)$, where $p = \gamma n^{-1/\delta(H)} = \gamma n^{-k/l}$ for some $\gamma = \gamma_n$. Let us first apply the first-moment method to show that if $\gamma \to 0$, then almost no G contains a subgraph isomorphic to H. Denote

$$\mathcal{H} = \{\text{all copies of } H \text{ on vertex-set of } G\}.$$

Then $|\mathcal{H}| = \binom{n}{k} h \leq \binom{n}{k} k! \leq n^k$, where h is the number of different arrangements of H on a set of k vertices, $h = k!/|\mathrm{Aut}(H)|$. Thus

$$\begin{aligned} E[X] &= \sum_{H' \in \mathcal{H}} \Pr(H' \subseteq G) = |\mathcal{H}| \cdot p^l \\ &\leq n^k p^l = n^k (\gamma n^{-k/l})^l = \gamma^l \xrightarrow[\gamma \to 0]{} 0, \end{aligned}$$

and by Markov's inequality the desired result follows.

For the other part, we wish apply the second-moment method to show that if $\gamma \to \infty$, then almost every graph G contains a subgraph isomorphic to H. For this, we need to verify that $\mu = E[X] > 0$ for all sufficiently large n, and then show that

$$\frac{\sigma^2}{\mu^2} = \frac{1}{\mu^2}(E[X^2] - \mu^2) \to 0 \quad \text{as } n \to \infty.$$

The first condition is easy to check: without loss of generality, assume that $\gamma = \gamma_n \geq 1$ for all n. Then:

$$\begin{aligned} \mu &= E[X] = |\mathcal{H}| \cdot p^l \\ &= \binom{n}{k} h \cdot \gamma_n^l \cdot n^{-k} \\ &\geq \text{const} \cdot n^k \cdot h \cdot \gamma_n^l \cdot n^{-k} \\ &> 0. \end{aligned}$$

For the other requirement, let us try to compute:

$$
\begin{aligned}
E[X^2] &= \sum_{H',H''\in\mathcal{H}} \Pr(H'\cup H''\subseteq G) \\
&= \sum_{H',H''\in\mathcal{H}} p^{e(H')+e(H'')-e(H'\cap H'')} \\
&\le \sum_{H',H''\in\mathcal{H}} p^{2l-i\delta(H)},
\end{aligned}
$$

where $i = |H'\cap H''|$. (Note that $\delta(H'\cap H'') \le \delta(H)$.)
Denote then $\mathcal{H}_i^2 = \{(H',H'') \in \mathcal{H}^2 : |H'\cap H''| = i\}$ and compute separately for each i the sum

$$
A_i = \sum_{\mathcal{H}_i^2} \Pr(H'\cup H''\subseteq G)
$$

Case $i = 0$*:*

$$
\begin{aligned}
A_0 &= \sum_{\mathcal{H}_0^2} \Pr(H'\cup H''\subseteq G) \\
&= \sum_{\mathcal{H}_0^2} \Pr(H'\subseteq G)\cdot\Pr(H''\subseteq G) \qquad H',H'' \text{ independent} \\
&\le \sum_{\mathcal{H}^2} \Pr(H'\subseteq G)\cdot\Pr(H''\subseteq G) \\
&= \left(\sum_{\mathcal{H}} \Pr(H'\subseteq G)\right)^2 \\
&= \mu^2.
\end{aligned}
$$

Case $i \geq 1$:

$$
\begin{aligned}
A_i &= \sum_{\mathcal{H}_i^2} \Pr(H' \cup H'' \subseteq G) \\
&= \sum_{H' \in \mathcal{H}} \sum_{\substack{H'': \\ |H' \cap H''| = i}} \Pr(H' \cup H'' \subseteq G) \\
&\leq |\mathcal{H}| \cdot \binom{k}{i} \binom{n-k}{k-i} h p^{2l} p^{-il/k} \qquad\qquad h = \frac{k!}{|\mathrm{Aut}(H)|} \\
&\leq |\mathcal{H}| \cdot c_1 n^{k-i} h p^{2l} (\gamma n^{-k/l})^{-il/k} \\
&= \mu \cdot c_1 n^{k-i} h p^l \gamma^{-il/k} n^i \\
&= \mu \cdot c_1 n^k h p^l \gamma^{-il/k} \\
&= \mu c_2 \underbrace{\binom{n}{k} h\, p^l}_{|\mathcal{H}|} \gamma^{-il/k} \\
&= \mu^2 \cdot c_2 \gamma^{-il/k} \\
&\leq \mu^2 \cdot c_2 \gamma^{-l/k}.
\end{aligned}
$$

Thus, denoting $c_3 = kc_2$, we get the estimate

$$\frac{E[X^2]}{\mu^2} = \left(\frac{A_0}{\mu^2} + \frac{\sum_i A_i}{\mu^2} \right) \leq 1 + c_3 \gamma^{-l/k}$$

and hence

$$\frac{\sigma^2}{\mu^2} = \frac{E[X^2] - \mu^2}{\mu^2} \leq c_3 \gamma^{-l/k} \xrightarrow[\gamma \to \infty]{} 0.$$

The desired result then follows by Lemma 7.10. □

Corollary 7.12 *For $k \geq 3$, the property of containing a k-cycle has threshold $t(n) = n^{-1}$. (Note that the threshold is independent of k.)*□

Corollary 7.13 *For $k \geq 2$, the property of containing a specific tree structure T on k nodes has threshold function $t(n) = n^{-k/(k-1)}$.* □

Corollary 7.14 *For $k \geq 2$, the property of containing a k-clique ($\approx K_k$) has threshold function $t(n) = n^{-2/(k-1)}$.* □

Denote $\delta^*(H) = \max\{\delta(H') | H' \text{ is subgraph of } H\}$.

Theorem 7.11' *The graph property "G has a subgraph isomorphic to H" has threshold function $n^{-1/\delta^*(H)}$.* □