T-79.7001 Postgraduate Course in Theoretical Computer Science - Spring 2006

Thomson's principle and Rayleigh's monotonicity law

Martti Meri

- Currents minimize energy dissipation
- Conservation of energy
- Thomson's principle
- Rayleigh's monotonicity law
- A probabilistic explanation of the Monotonicity law
- A Markov chain proof of the Monotonicity law

Currents minimize energy dissipation

$$E = \sum_{x,y} i_{xy}^2 R_{xy} = \sum_{x,y} i_{xy}(v_x - v_y) \tag{1}$$

$j_a + j_b = \sum_x j_x = \sum_x \sum_y j_{xy} =$
$\frac{1}{2} \sum_{x,y}(j_{x,y} - j_{y,x}) = 0$

$\sum_{x,y}(w_x - w_y)j_{xy} = \sum_x(w_x \sum_y j_{xy}) - \sum_y(w_y \sum_x j_{xy}) =$
$w_a \sum_y j_{ay} + w_b \sum_y j_{by} - w_a \sum_x j_{xa} - w_b \sum_x j_{xb} =$
$w_a j_a + w_b j_b - w_a(-j_a) - w_b(-j_b) = 2(w_a - w_b)j_a$

Conservation of energy

$$(w_a - w_b)j_a = \frac{1}{2}\sum_{x,y}(w_x - w_y)j_{xy} \tag{2}$$

$$v_a i_a = \frac{1}{2}\sum_{x,y}(v_x - v_y)i_{xy} = \frac{1}{2}\sum_{x,y} i_{xy}^2 R_{x,y} \tag{3}$$

$R_{eff} = v_a/i_a$

$$i_a^2 R_{eff} = \frac{1}{2}\sum_{x,y} i_{xy}^2 R_{x,y} \tag{4}$$

energy dissipated by unit current flow is just R_{eff}

Thomson's principle

If i *is the unit flow from* a *to* b *determined by Kirchhoff's Laws, then the energy dissipation* $\sum_{x,y} i_{xy}^2 R_{xy}$ *minimizes the energy dissipation* $\sum_{x,y} j_{xy}^2 R_{xy}$ *among all unit flows* j *from* a *to* b.

$d_{x,y} = j_{x,y} - i_{x,y}$. d is a flow from a to b with
$d_a = \sum_x d_{a,x} = 1 - 1 = 0$.
$\sum_{x,y} j_{xy}^2 R_{xy} = \sum_{x,y} (i_{xy} + d_{x,y})^2 R_{xy} =$
$\sum_{x,y} i_{xy}^2 R_{xy} + 2\sum_{x,y} i_{xy} R_{xy} d_{x,y} + \sum_{x,y} d_{xy}^2 R_{xy} =$
$\sum_{x,y} i_{xy}^2 R_{xy} + 2\sum_{x,y} (v_x - v_y) d_{x,y} + \sum_{x,y} d_{xy}^2 R_{xy}$ setting
$w = v$ and $j = d$ and recalling (2.) the middle term is
$4(v_a - v_b) d_a = 0$

$$\sum_{x,y} j_{xy}^2 R_{xy} = \sum_{x,y} i_{xy}^2 R_{xy} + \sum_{x,y} d_{xy}^2 R_{xy} \geq \sum_{x,y} i_{xy}^2 R_{xy} \ .$$

Rayleigh's monotonicity law

If the resistances of a circuit are increased, the effective resistance R_{eff} between any two points can only increase. If the are decreased, it can only decrease.

i unit current from a to b with the values $R_{x,y}$ and j unit current from a to b with the values $\overline{R}_{x,y}$ so that $\overline{R}_{x,y} \geq R_{x,y}$.

$\overline{R}_{eff} = \frac{1}{2}\sum_{x,y} j_{xy}^2 \overline{R}_{x,y} \geq \frac{1}{2}\sum_{x,y} j_{xy}^2 R_{x,y} \geq \frac{1}{2}\sum_{x,y} i_{xy}^2 R_{x,y} = R_{eff}$

In the last inequation Thomson's principle used.

A probabilistic explanation of the Monotonicity law

Assume $v_r > v_s$ and mark u_x expected number of times in x and u_{xy} expected number of crossing from x to y.

$$u_{rs} = u_r P_{rs} = u_r \frac{C_{rs}}{C_r} = v_r C_{rs}$$

$$u_{sr} = u_s P_{sr} = u_s \frac{C_{sr}}{C_s} = v_s C_{sr}$$

Since $C_{rs} = C_{sr}$ and recalling assumption we have $u_{rs} > u_{sr}$

Effect on escape probability

Claim. $p_{esc}^{(\epsilon)} = p_{esc} + (v_r - v_s)d^{(\epsilon)}$ *where* $d^{(\epsilon)} = u_r^{(\epsilon)}\frac{\epsilon}{C_r+\epsilon} - u_s^{(\epsilon)}\frac{\epsilon}{C_s+\epsilon}$ *denotes the expected net number of times the walker crosses from* r *to* s

Proof. Fortune at x is v_x is the probability of returning to a before reaching b in the absence of bridge rs.

$1 \cdot (1 - p_{esc}^{(\epsilon)}) + 0 \cdot p_{esc}^{(\epsilon)} = 1 - p_{esc}^{(\epsilon)}$

lost amount when stepping away from r.

$v_r - \left(\sum_x \frac{C_{rx}}{C_r+\epsilon} v_x + \frac{\epsilon}{C_r+\epsilon} v_s \right) = (v_r - v_s)\frac{\epsilon}{C_r+\epsilon}$

total amount expected to loose

$p_{esc} + (v_r - v_s)\frac{\epsilon}{C_r+\epsilon} + (v_s - v_r)\frac{\epsilon}{C_s+\epsilon} = p_{esc} + (v_r - v_s)d^{(\epsilon)}$

for small ϵ

$$\frac{u_r^{(\epsilon)}}{C_r+\epsilon} - \frac{u_s^{(\epsilon)}}{C_s+\epsilon} = \frac{u_r}{C_r} - \frac{u_s}{C_s} = \frac{v_r}{C_a} - \frac{v_s}{C_a}$$

meaning that,

$$p_{esc}^{(\epsilon)} - p_{esc} = (v_r - v_s)^2 \frac{\epsilon}{C_a}$$

monotonicity law

$$p_{esc}^{(\epsilon)} \geq p_{esc} \geq 0$$

A Markov chain proof of the Monotonicity law

P is ergodic Markov chain associated with an electric network.

Bridge ϵ from r to s. $P^{(\epsilon)}_{rs} = \frac{C_{rs}+\epsilon}{C_r+\epsilon}$ $\qquad \hat{P}_{rs} = \frac{C_{rs}}{C_r+\epsilon}$

$P^{(\epsilon)}_{rr} = 0 \qquad \hat{P}_{rr} = \frac{\epsilon}{C_r+\epsilon}$

$$Q^{(\epsilon)} = \hat{Q} + hk \tag{5}$$

$$h = \left(0, \ldots, 0, (\frac{\epsilon}{C_r + \epsilon}), 0, \ldots 0, (-\frac{\epsilon}{C_s + \epsilon}), 0, \ldots, 0\right)^T$$

$$k = (0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, \ldots, 0)$$

$$N^{(\epsilon)} = \hat{N} + c(\hat{N}h)(k\hat{N}) \tag{6}$$

$$N^{(\epsilon)}_{i,j} = \hat{N}_{i,j}\epsilon + \Big(\frac{\hat{N}_{i,r}\epsilon}{C_r + \epsilon} - \frac{\hat{N}_{i,s}\epsilon}{C_s + \epsilon}\Big)(\hat{N}_{sj} - \hat{N}_{rj}) \tag{7}$$

$$c = \frac{1}{1 + \frac{\hat{N}_{r,r}\epsilon}{C_r+\epsilon} - \frac{\hat{N}_{s,r}\epsilon}{C_r+\epsilon} + \frac{\hat{N}_{s,s}\epsilon}{C_s+\epsilon} - \frac{\hat{N}_{r,s}\epsilon}{C_s+\epsilon}} \tag{8}$$

$$B_{xb} = \sum_y N_{x,y} P_{y,b} \tag{9}$$

since $P^{(\epsilon)}_{x,b} = \hat{P}_{x,b}$

$$B_{xb}^{(\epsilon)} = \hat{B}_{xb} + c(\frac{\hat{N}_{x,r}\epsilon}{C_r + \epsilon} - \frac{\hat{N}_{x,s}\epsilon}{C_s + \epsilon})(\hat{B}_{sb} - \hat{B}_{rb}) \tag{10}$$

since $P_{a,x}^{(\epsilon)} = \hat{P}_{a,x}$

$$p_{esc}^{(\epsilon)} = \hat{p}_{esc} + c(\frac{\hat{u}_r\epsilon}{C_r + \epsilon} - \frac{\hat{u}_s\epsilon}{C_s + \epsilon})(\hat{B}_{sb} - \hat{B}_{rb}) \tag{11}$$

$\frac{u_x}{\hat{C}_x} = \frac{\hat{B}_{x,a}}{\hat{C}_a} = \frac{\hat{B}_{x,a}}{C_a}$

$$p_{esc}^{(\epsilon)} = p_{esc} + \frac{\epsilon c}{C_a}(\hat{B}_{sb} - \hat{B}_{rb})^2 \tag{12}$$