T-79.7001 Postgraduate Course in Theoretical Computer Science - Spring 2006

Vertex-partitions and the spectrum (Biggs sec. 8)

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Definitions colour-partition V_i: $V\Gamma = V_1 \cup V_2 \cup \ldots \cup V_l$, so that each $V_i (1 \leq i \leq l)$ contains no pair of adjacent vertices.

chromatic number $\nu(\Gamma)$ is the least natural number l for which such partition is possible.

vertex-colouring: assignment of vertex colours with adjacent vertices having different colour. Vertex-colouring with l colours gives rise to a colour partition with l colour-classes.

A graph is l-critical if $\nu(\Gamma) = l$ and for all induced subgraphs $\Lambda \neq \Gamma$ we have $\nu(\Lambda) < l$.

Lemma The chromatic number of a graph X is the least integer l such that there is a homomorphism from X to K_l. (Codsil and Royle)

A l-critical graph cannot have a homomorphism to any proper subgraph, and hence must be its own core. This provides a wide class of cores, including all complete graphs and odd cycles. (Codsil and Royle)

The Four Colour Theorem Every planar graph has a vertex-colouring with $l = 4$ colours

Definitions (cont.)

Rayleigh quotient: $R(X;z) = \langle z, Xz\rangle / \langle z, z\rangle$

maximum and minimum eigenvalues of X $\lambda_{max}(X)$ and $\lambda_{min}(X)$

$\lambda_{max}(X) \geq R(X;z) \geq \lambda_{min}(X)$,for all $z \neq 0$.

Proof: $U^t X U = D$. Substituting $X = UDU^t$ and $z = Uy$.
$R(X;y) = \langle y, Dy\rangle / \langle y, y\rangle = y^t Dy / y^t (U^t U) y = \sum_i \lambda_i |y_i|^2 / \sum_i |y_i|^2$ so the equations above hold.

Proposition 8.3

(1) If Λ is an induced subgraph of Γ, then
$\lambda_{max}(\Lambda) \leq \lambda_{max}(\Gamma); \lambda_{min}(\Lambda) \geq \lambda_{min}(\Gamma)$.

(2) If the greates and least degrees among the vertices of Γ are $k_{max}(\Gamma)$ and $k_{min}(\Gamma)$, and the average degree is $k_{ave}(\Gamma)$, then
$k_{max}(\Gamma) \geq \lambda_{max}(\Gamma) \geq k_{ave}(\Gamma) \geq k_{min}(\Gamma)$.

Lemma 8.4 Suppose Γ is a graph with chromatic number $l \geq 2$. Then Γ has a l-critical induced subgraph Λ, and every vertex of Λ has degree at least $l-1$ in Λ.

Proof:The set of induced subgraphs of Γ is non-empty and contains graphs with ν of l and graphs with ν of not l . Let Λ be an induced subgraph with $\nu(\Lambda) = l$, and with minimal $|V\Lambda|$, then Λ is l-critical.

Since if Λ were not l-critical, there would be a induced subgraph $\Upsilon \neq \Lambda$ of Λ with $\nu(\Upsilon) \geq l$. But this means that Λ is not minimal in size of the induced subgraphs of Γ with $\nu(\Lambda) = l$.

If $v \in \Lambda$, then $\langle V\Lambda \setminus v \rangle$ is an induced subgraph of Λ and has a vertex-colouring with $l-1$ colours. If the degree of v in Λ is less than $l-1$, then we could extend this vertex-colouring to Λ, contradicting the fact that $\nu(\Lambda) = l$. Thus the degree of v is at least $l-1$.

Proposition 8.5 For any graph Γ we have $\nu(\Gamma) \leq 1 + \lambda_{max}(\Gamma)$.

Lemma 8.6 Let X be a real symmetric matrix, partitioned in the form

$$\mathbf{A} = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix}$$

, where P and Q are square symmetric matrices. Then
$\lambda_{max}(X) + \lambda_{min}(X) \leq \lambda_{max}(P) + \lambda_{max}(R)$.

Corollary 8.7 Let A be a real symmetric matrix, partitioned into t^2 submatrices A_{ij} in such a way that the row and column partitions are the same; in other words, each diagonal sub-matrix $A_{ii}(1 \leq i \leq t)$ is square, then

$\lambda_{max}(A) + (t-1)\lambda_{min}(A) \leq \sum_{i=0}^{t} \lambda_{max}(A_{ii})$.

Theorem 8.8

For any graph Γ , whose edge set is non-empty,

$\nu(\Gamma) \geq 1 + \frac{\lambda_{max}(\Gamma)}{-\lambda_{min}(\Gamma)}$.