Isoperimetric problems

Satu Elisa Schaeffer

elisa.schaeffer@tkk.fi

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The isoperimetric problem

Among all **closed curves** of length ℓ, which one encloses the **maximum area**?

For graphs: separator problems (vertex and **edge cuts**) — relations between the cut sizes and the sizes of the separated parts

Volume and boundary

- *Notation*: graph $G = (V, E(G))$, set $S \subset V$, $|V| = n$
- **Volume**: $\operatorname{vol} S = \sum_{v \in S} d_v$
- **Edge boundary**: $\partial S = \{\{u, v\} \in E(G) \mid u \in S, v \notin S\}$
- **Vertex boundary**: $\delta S = \{v \notin S \mid \{u, v\} \in E(G), u \in S\}$

RELATED PROBLEMS

Given a fixed integer m, find a subset S with $m \leq \text{vol}\, S \leq \text{vol}\, \bar{S}$ s.t.

1. the boundary $\partial S = \{\{u, v\} \in E(G) \mid u \in S, v \notin S\}$ contains as *few edges* as possible
2. the boundary $\delta S = \{v \notin S \mid \{u, v\} \in E(G), u \in S\}$ contains as *few vertices* as possible

Cheeger constant

$$h_G = \min_S \frac{|\partial S|}{\min\left\{\mathsf{vol}\, S, \mathsf{vol}\, \bar{S}\right\}}$$

From the definition, we get for S s.t. vol $S <$ vol $\bar{S}$ that $|\partial S| \geq h_G \cdot$ vol S.

Also, G is connected iff $h_G > 0$.

Vertex expansion

$$g_G = \min_S \frac{|\delta S|}{\min\{\text{vol}\, S, \text{vol}\, \bar{S}\}}, \quad \textbf{Regular}\text{ graphs: } g_G(S) = \frac{|\delta S|}{\min\{|S|, |\bar{S}|\}}$$

Definition: (volume replaced by unit measure)

$$\bar{g}_G = \min_S \frac{|\delta S|}{\min\{|S|, |\bar{S}|\}}$$

LEMMA: $2h_G \geq \lambda_1$

Setup for the proof:

- C is a cut that achieves h_G
- C splits V into sets A and B
- Definition: $f(v) = \begin{cases} \dfrac{1}{\text{vol}\, A}, & \text{if } v \in A, \\ -\dfrac{1}{\text{vol}\, B}, & \text{if } v \in B \end{cases}$

Expression for λ_1

$$\lambda_1 = \lambda_G = \inf_{f \perp T_1} \frac{\sum_{u \sim v} \left(f(u) - f(v)\right)^2}{\sum_{v} \left(f(v)\right)^2 d_v}$$

PROOF OF $2h_G \geq \lambda_1$, PART $\frac{1}{2}$

Using the definition of λ_1 with definitions of vol S, C and f, we get the result. First we simply “partition” the expression using A and B:

$$\lambda_1 = \inf_{f \perp T_1} \frac{\sum_{u \sim v} (f(u) - f(v))^2}{\sum_{v} (f(v))^2 \, d_v}$$

$$= \frac{\sum_{\substack{u \in A, \\ v \in B}} (f(u) - f(v))^2 + \sum_{\substack{u \in A, \\ v \in A}} (f(u) - f(v))^2 + \sum_{\substack{u \in B, \\ v \in B}} (f(u) - f(v))^2}{\sum_{v \in A} (f(v))^2 d_v + \sum_{v \in B} (f(v))^2 d_v}$$

PROOF CONTINUES, PART $\frac{2}{2}$

We use the definitions of f and vol:

$$\lambda_1 = \frac{\sum_{u \in A, v \in B} \left(\frac{1}{\text{vol}\, A} + \frac{1}{\text{vol}\, B} \right)^2 + 0 + 0}{\sum_{v \in A} \frac{d_v}{(\text{vol}\, A)^2} - \sum_{v \in B} \frac{d_v}{(\text{vol}\, B)^2}}$$

$$= \frac{|C| \left(\frac{1}{\text{vol}\, A} + \frac{1}{\text{vol}\, B} \right)^2}{\frac{1}{(\text{vol}\, A)^2} \cdot \text{vol}\, A + \frac{1}{(\text{vol}\, B)^2} \cdot \text{vol}\, B}$$

$$= |C| \left(\frac{1}{\text{vol}\, A} + \frac{1}{\text{vol}\, B} \right)$$

THEOREM: $\lambda_1 > \frac{h_G^2}{2}$

Setup for proof:

- vertex **labels** $v_1, v_2, \ldots, v_n$ such that $f(v_i) \leq f(v_{i+1})$ $(1 \leq i \leq n-1)$
- w.l.o.g. $\sum_{f(v)<0} d_v \geq \sum_{f(v)\geq 0} d_v$
- **cuts** $C_i = \{\{v_j, v_k\} \in E(G) \mid 2 \leq j \leq i < k \leq n\}$, $1 \leq i \leq n$

DEFINITIONS FOR THE PROOF

- *Definition*: $\alpha = \min\limits_{1 \le i \le n} \dfrac{|C_i|}{\min\left\{\sum\limits_{j \le i} d_j, \sum\limits_{j > i} d_j\right\}}$
- By definition $\alpha \ge h_G$ (divisors are the volumes of the parts)
- $V_+ = \{v \in V \mid f(v) \ge 0\}$
- $E_+ = \big\{\{u, v\} \in E(G) \mid u \in V_+, v \in V\big\}$
- $g(v) = \begin{cases} f(v), & \text{if } v \in V_+, \\ 0, & \text{otherwise} \end{cases}$

HARMONIC EIGFN f OF $\mathcal{L}$ WITH EIGVAL λ_1

For any $v \in V$, it holds for f that

$$\frac{1}{d_v} \sum_{u \sim v} \big(f(v) - f(u)\big) = \lambda_1 f(v)$$

$$\Rightarrow \lambda_1 = \frac{1}{d_v f(v)} \sum_{u \sim v} \big(f(v) - f(u)\big) \quad (\dagger)$$

(a lemma from the previous chapter)

PROOF OF THE THEOREM, PART $\frac{1}{8}$

Substituting $\lambda_1 = (\dagger)$ and summing over V_+

$$\lambda_1 = \frac{1}{d_v f(v)} \sum_{u \sim v} \big(f(v) - f(u)\big) \qquad (\dagger)$$

$$= \frac{\displaystyle\sum_{v \in V_+} f(v) \sum_{\{u,v\} \in E_+} \big(f(v) - f(u)\big)}{\displaystyle\sum_{v \in V_+} \big(f(v)\big)^2 d_v} \qquad (\triangle)$$

because for any subset $S \subseteq V$, we have

$$\lambda_1 f(v) d_v = \sum_{u \sim v} \big(f(v) - f(u)\big)$$

$$\lambda_1 (f(v))^2 d_v = f(v) \sum_{u \sim v} \big(f(v) - f(u)\big)$$

$$\lambda_1 \sum_{v \in S} (f(v))^2 d_v = \sum_{v \in S} f(v) \sum_{u \sim v} \big(f(v) - f(u)\big)$$

Proof of the theorem, part $\frac{2}{8}$

From the defs of g, V_+ and E_+ (as $(f(u))^2 > 0$ and $g(v) \geq f(v)$),

$$\begin{aligned}
\lambda_1 &= \frac{\sum_{v \in V_+} f(v) \sum_{\{u,v\} \in E_+} \big(f(v) - f(u)\big)}{\sum_{v \in V_+} \big(f(v)\big)^2 d_v} \qquad (\triangle) \\
&= \frac{\sum_{\substack{v \in V_+ \\ \{u,v\} \in E_+}} \big((f(v))^2 - f(v)f(u)\big)}{\sum_{v \in V_+} \big(f(v)\big)^2 d_v} \\
&> \frac{\sum_{\{u,v\} \in E_+} \big(g(u) - g(v)\big)^2}{\sum_{v \in V} \big(g(v)\big)^2 d_v} \qquad (*)
\end{aligned}$$

Proof of the theorem, part $\frac{3}{8}$

Using the *Cauchy-Schwarz inequality* $(\sum x_i y_i)^2 \le (\sum x_i^2)(\sum y_i^2)$ with $x_i = |g(u) - g(v)|$ and $y_i = g(u) + g(v)$, we get

$$\lambda_1 > \frac{\sum_{\{u,v\}\in E_+} \big(g(u)+g(v)\big)^2}{\sum_{\{u,v\}\in E_+} \big(g(u)+g(v)\big)^2} \cdot \frac{\sum_{\{u,v\}\in E_+} \big(g(u)-g(v)\big)^2}{\sum_{v\in V} \big(g(v)\big)^2 d_v} \qquad (*)$$

$$\geq \frac{\sum_{u\sim v} |\big(g(u)-g(v)|\big)\big(g(u)+g(v)\big)}{2\left(\sum_v \big(g(v)\big)^2 d_v\right)^2}$$

Proof of the theorem, part $\frac{4}{8}$

Now using $(a+b)(a-b) = a^2 - b^2$, we get

$$\lambda_1 \geq \frac{\sum_{u\sim v} |\big(g(u)-g(v)|\big)\big(g(u)+g(v)\big)}{2\left(\sum_v \big(g(v)\big)^2 d_v\right)^2}$$

$$\geq \frac{\left(|\big(g(u)\big)^2 - \big(g(v)\big)^2|\right)^2}{2\left(\sum_v \big(g(v)\big)^2 d_v\right)^2}$$

Proof of the theorem, part $\frac{5}{8}$

Now from the definition of C_i and "partitioning" the edges to "steps" over the cuts C_i, we continue

$$\begin{aligned}
\lambda_1 &\geq \frac{\left(\sum_{u\sim v} \left|\big(g(u)\big)^2 - \big(g(v)\big)^2\right|\right)^2}{2\left(\sum_{v} \big(g(v)\big)^2 d_v\right)^2} \\
&= \frac{\left(\sum_{i} |(g(v_i))^2 - (g(v_{i+1}))^2| \cdot |C_i|\right)^2}{2\left(\sum_{v} (g(v))^2 d_v\right)^2}.
\end{aligned}$$

Proof of the theorem, part $\frac{6}{8}$

Using the definition of α together with the fact that $\sum_{f(v)<0} d_v \geq \sum_{f(v)\geq 0} d_v$ and the vertex ordering, we get

$$\begin{aligned}
\lambda_1 &\geq \frac{\left(\sum_i |(g(v_i))^2 - (g(v_{i+1}))^2| \cdot |C_i|\right)^2}{2\left(\sum_v (g(v))^2 d_v\right)^2} \\
&\geq \frac{\left(\sum_i (g(v_i))^2 - (g(v_{i+1}))^2 \cdot \alpha \sum_{j>i} d_j\right)^2}{2\left(\sum_v (g(v))^2 d_v\right)^2}.
\end{aligned}$$

Proof of the theorem, part $\frac{7}{8}$

$$\frac{\left(\sum_i (g(v_i))^2 - (g(v_{i+1}))^2 \sum_{j>i} d_j\right)^2}{\left(\sum_v (g(v))^2 d_v\right)^2} = \frac{\sum_{i=0}^{n-1} \big(g(v_{i+1})\big)^2 d_{i+1}}{\sum_{v=1}^{n} \big(g(v)\big)^2 d_v} = 1$$

as when we multiply the nominator “open”, all but one of the $\big(g(v_{i+1})\big)^2$ cancel out, appearing both positive and negative, except for once for $j = i + 1$, which leaves the same summation than we have in the denumerator.

PROOF OF THE THEOREM, PART $\frac{8}{8}$

Now we simply take out α^2 and use the previous observation and the definition of α to complete the proof:

$$\lambda_1 \quad \geq \quad \frac{\left(\sum_i (g(v_i))^2 - (g(v_{i+1}))^2 \cdot \alpha \sum_{j>i} d_j\right)^2}{2\left(\sum_v (g(v))^2 d_v\right)^2} = \frac{\alpha^2}{2} \geq \frac{h_G^2}{2}.$$

CHEEGER INEQUALITY

Putting together the lemma and the theorem, we have

$$2h_G \geq \lambda_1 > \frac{h_G^2}{2}.$$

IMPROVEMENT: $\lambda_1 > 1 - \sqrt{1 - h_G^2}$

From the proof of the previous theorem we have $\lambda_1 = (\triangle)$ and we define $W = (*)$:

$$
\begin{aligned}
\lambda_1 &= \frac{\displaystyle\sum_{v \in V_+} f(v) \sum_{\{u,v\} \in E_+} \big(f(v) - f(u)\big)}{\displaystyle\sum_{v \in V_+} \big(f(v)\big)^2 d_v} \\
&> \frac{\displaystyle\sum_{\{u,v\} \in E_+} \big(g(u) - g(v)\big)^2}{\displaystyle\sum_{v \in V} \big(g(v)\big)^2 d_v} = W
\end{aligned}
$$

PROOF OF THE SECOND THEOREM

Again we extend and use some already familiar tricks (plugging in the def. of W itself):

$$
\begin{aligned}
W &= \frac{\sum_{\{u,v\}\in E_+} \big(g(u)+g(v)\big)^2}{\sum_{\{u,v\}\in E_+} \big(g(u)+g(v)\big)^2} \cdot \frac{\sum_{\{u,v\}\in E_+} \big(g(u)-g(v)\big)^2}{\sum_{v\in V} \big(g(v)\big)^2 d_v} \\
&\geq \frac{\left(\sum_{u\sim v} |(g(u))^2-(g(v))^2|\right)^2}{\left(\sum_v (g(v))^2 d_v\right)\cdot\left(2\sum_v (g(v))^2 d_v - W\sum_v (g(v))^2 d_v\right)}
\end{aligned}
$$

PROOF CONTINUES

Rewriting the nominator just as in the previous proof, simple factorization of the denominator gives

$$
\begin{aligned}
W &\geq \frac{\left(\sum_i |(g(v_i))^2 - (g(v_{i+1}))^2| \cdot |C_i|\right)^2}{(2-W)\left(\sum_v (g(v))^2\right)^2 d_v} \\
&\geq \frac{\left(\sum_i |(g(v_i))^2 - (g(v_{i+1}))^2| \cdot \alpha \sum_{j>i} d_j\right)^2}{(2-W)\left(\sum_v (g(v))^2\right)^2 d_v} \\
&= \frac{\alpha^2}{2-W}
\end{aligned}
$$

INTERMEDIATE RESULT: $W \geq \frac{\alpha^2}{2-W}$

$$\Rightarrow W^2 - 2W + \alpha^2 \leq 0.$$

Solving the zeroes gives $W \geq 1 - \sqrt{1-\alpha^2}$.

By definitions of W and α, we have $\lambda_1 > W$ and $\alpha \geq h_G$. Hence **we have proved the theorem** $\lambda_1 > 1 - \sqrt{1-h_G^2}$. Note that

$$\frac{h_G^2}{2} < 1 - \sqrt{1-h_G^2}$$

whenever $h_G > 0$ (i.e., for any connected graph), meaning that this is always an improvement to the previous lower bound.

CONSTRUCTIONAL "COROLLARY"

In a graph G with eigfn f associated with λ_1, define for each $v \in V$

$$C_v = \big\{\{u, w\} \in E(G) \mid f(u) \le f(v) < f(w)\big\}$$

and

$$\alpha = \min_v |C_v| \cdot \min \left\{ \sum_{\substack{u \\ f(u) \le f(v)}} d_u, \sum_{\substack{u \\ f(u) > f(v)}} d_u \right\}^{-1}.$$

Then $\lambda_1 > 1 - \sqrt{1 - \alpha^2}$.

LOWER BOUND ON λ_1

For a connected simple graph G, $h_G \geq \dfrac{2}{\operatorname{vol} G}$.

From Cheegers inequality, $2h_G \geq \lambda_1 > \dfrac{h_G^2}{2}$, we have

$$\lambda_1 > \frac{1}{2}\left(\frac{2}{\operatorname{vol} G}\right)^2.$$

As $\operatorname{vol} G = 2|E(G)| \leq n(n-1) \leq n^2$, we get a lower bound

$$\lambda_1 \geq \frac{2}{n^4}.$$