Gentzen Sequent Calculus LK

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Completeness of Gentzen LK

- Start from the root $\Gamma \mapsto \Delta$
- Given a node $\Phi \mapsto \Psi$, break both Φ and Ψ until they only contain propositional variables
  - Can be done using the rules of inference
  - The cut rule is not needed
- Prove that after having been broken to propositional variables, $\Phi \cap \Psi \neq \emptyset$ if $\models \Gamma \rightarrow \Delta$
- For each such $\Phi \mapsto \Psi$, select a propositional variable $p \in \Phi \cap \Psi$, and add the sequent $p \mapsto p$ (using weakening)

Notes to the proof

The proof is constructive, in the sense that it gives a method for constructing any proof for a tautology $\Gamma \to \Delta$

- The cut rule was not used
  - It is not neeed for completeness
- Proof is tree-like
- Cut-free tree-like Gentzen sequent calculus is equivalent to *analytical tableaxu*
  - Widely used in machine theorem proving
  - For more information, see T-79.3001 (4cr)
- The size of the proof is $2^{\mathcal{O}(n)}$

Subformula property

A proof has the *subformula property*, if every formula appearing in every sequent of the proof is a subformula of a formula appearing in the root of the proof

- Subformula property is beneficial for machine theorem proving
- However, it has a cost showing in the length of the proof
  - The complexity of the previous proof is $2^{\mathcal{O}(n)}$
- In following, we prove that
  - There is a tree-like proof with cut of length $\mathcal{O}(n^2)$ for a given tautology $\Gamma_n \mapsto \Delta_n$.
  - Every tree-like cut-free proof of the same tautology has at least length 2^n

Inference rules for propositional implication

In formulating the proofs, we need two additional rules of inference, which concern the implication in propositional logic.

$$\supset\text{-right}\frac{\phi, \Gamma \mapsto \psi, \Delta}{\Gamma \mapsto \phi \supset \psi, \Delta}$$

and

$$\supset\text{-left}\frac{\Gamma \mapsto \phi, \Delta \quad \Gamma, \psi \mapsto \Delta}{\Gamma, \phi \supset \psi \mapsto \Delta}$$

Defining $\Gamma_n \mapsto \Delta_n$

- Define ϕ_i as

$$\bigwedge_{j=1}^{i} (p_j \vee q_j)$$

- Define α_1 as p_1 and β_1 as q_1, and for $2 \leq i \leq n$, α_i as

$$\left(\bigwedge_{j=1}^{i-1} (p_j \vee q_j) \right) \supset p_i,$$

  and β_i as

$$\left(\bigwedge_{j=1}^{i-1} (p_j \vee q_j) \right) \supset q_i$$

- define Γ_i as $\{\alpha_1 \vee \beta_1, \ldots, \alpha_i \vee \beta_i\}$
- define Δ_i as $\{p_i, q_i\}$

$\Gamma_n \mapsto \Delta_n$ for $n = 1, 2, 3$

- for $n = 1$, we have

$$p_1 \vee q_1 \mapsto p_1, q_1$$

- for $n = 2$, we have

$$p_1 \vee q_1, (p_1 \vee q_1 \supset p_2) \vee (p_1 \vee q_1 \supset q_2) \mapsto p_2, q_2$$

- for $n = 3$, we have

$$\begin{gathered} p_1 \vee q_1, (p_1 \vee q_1 \supset p_2) \vee (p_1 \vee q_1 \supset q_2), \\ ((p_1 \vee q_1) \wedge (p_2 \vee q_2) \supset p_3) \vee \\ ((p_1 \vee q_1) \wedge (p_2 \vee q_2) \supset q_3) \mapsto p_3, q_3 \end{gathered}$$

Tree-like proof with cut for $\Gamma_n \mapsto \Delta_n$

Proof is given in four steps

1. We prove that for $1 \leq i < n$, there is a tree-like cut-free proof for $\phi, \alpha_{i+1} \vee \beta_{i+1} \mapsto \phi_{i+1}$ of length $\mathcal{O}(n)$ and size $\mathcal{O}(n^2)$
2. We use 1 to give a proof that there is a tree-like proof of for $\Gamma_n \mapsto \phi_n$ with cut of length $\mathcal{O}(n^2)$ and size $\mathcal{O}(n^3)$
3. We show that there are tree-like cut-free proofs of $\phi_n \mapsto p_n, q_n$ of length $\mathcal{O}(n)$ and size $\mathcal{O}(n)$
4. Finally we show that from 2 and 3, we have a proof for $\Gamma_n \mapsto \Delta_n$

Lemma 1

There is a tree-like cut-free proof for $\phi \mapsto \phi$, of length $\mathcal{O}(|\phi|)$ and size $\mathcal{O}(|\phi|^2)$. This can be seen from the following

- There is one application of each of $\vee$-left, $\vee$-right, $\wedge$-left, $\wedge$-right, $\supset$-left, $\supset$-right, and $\neg$-left, $\neg$-right for each corresponding connective, resulting in length $\mathcal{O}(|\phi|)$
- For $\vee$-left, $\vee$-right, $\wedge$-left, $\wedge$-right, $\supset$-left, $\supset$-right, the tree divides in two at each application, but the size of the lines remain approximately same. This would mean approximately $\mathcal{O}(|\phi| \log |\phi|)$
- For $\neg$-left, $\neg$-right the size decreases at each application by one, resulting in worst-case behaviour $\mathcal{O}(|\phi|^2)$

Tree-like cut-free proof for $\phi_i, \alpha_{i+1} \vee \beta_{i+1} \mapsto \phi_{i+1}$

- Seen by directly applying the inference rules, which results in $\mathcal{O}(1)$ steps as three proofs of $\phi_i \mapsto \phi_i$ which is $3\mathcal{O}(|\phi_i|)$ number of proof steps and $3\mathcal{O}(|\phi_i|^2)$ size proof (by Lemma 1)
- Thus, we have the proof of length $\mathcal{O}(i)$ and size $\mathcal{O}(i^2)$
- By Lemma 1, this contradicts the proof given in the book. If we can prove tighter bounds for $\phi \mapsto \phi$, we could have the claim, perhaps.
- Anyway, some polynomial, so who cares?

Tree-like proof for $\Gamma_n \mapsto \phi_n$

- We prove that there is a tree-like proof for $\Gamma_n \mapsto \phi_n$ with cut having length $\mathcal{O}(n^2)$ and size $\mathcal{O}(n^3)$
  - Shown using cut and weakening to get sequents of the same form as in first claim. There will be linear number of them w.r.t. $|\Gamma_n|$, but since $|\Gamma_n| \in \mathcal{O}(n^2)$, we have that the length of the proof is $\mathcal{O}(n^2)$ and size is $\mathcal{O}(n^3)$.
- We prove that there are tree-like cut-free proofs of $\phi_n \mapsto p_n, q_n$ of length $\mathcal{O}(n)$ and size $\mathcal{O}(n)$
  - This is shown by using $\wedge$-left until only $p_n \vee q_n$ remain on the left side, and then using $\vee$-left and finally weakening to get the corresponding axioms

Combining proofs of $\Gamma_n \mapsto \phi_n$ and $\phi_n \mapsto p_n, q_n$

- Now we have proofs for $\Gamma_n \mapsto \phi_n$ and $\phi_n \mapsto p_n, q_n$. By using weakening and cut, we have $\Gamma_n \mapsto p_n, q_n$ which concludes the proof.

Tree-like cut-free proof for $\Gamma_n \mapsto \Delta_n$

We can prove that the smallest tree-like cut-free proof for the tautology is exponential in n by the following observations

- Shortest proof for $\phi \vee \psi, \Gamma \mapsto \Delta$ is always greater than the sum of the sequents $\phi, \Gamma \mapsto \Delta$ and $\psi, \Gamma \mapsto \Delta$ obtained by using $\vee$-left
- If the proof tree of $\Gamma_n \mapsto \Delta_n$ is minimal, then the last operation before root must have been a $\vee$-left.
- The two sequents are symmetrical, and each must have been constructed at some earlier point with $\supset$-left.
- The resulting sequent must have been constructed earlier with $\supset$-left, since there is no other way to introduce $\supset$ to the formula.
- Since p_n and q_n do not appear in any earlier formula, they must have been obtained by weakening

Case when $i \neq n$

The above method results in an exponential proof with regards to n, but fails if p_n or q_n appear earlier in the proof.

- Need to ensure that the same result holds even if $i \neq n$ is used as the last $\vee$-left

Given a fixed i, rewrite the tautology by using $\phi'_j, \alpha'_j, \beta'_j$ such that

$$\phi'_j \equiv \bigwedge_{1 \leq k \leq j, k \neq i} (p_k \vee q_k)$$

$$\alpha'_1 \equiv p_1$$

$$\beta_1 \equiv q_1$$

$$\alpha'_{j+1} \equiv \phi'_j \supset p_{j+1} \text{ for } j+1 \neq 1$$

$$\beta'_{j+1} \equiv \phi'_j \supset q_{j+1} \text{for } j+1 \neq 1$$

Now, by renaming the variables, we have a similar proof for $\Gamma_{n-1} \mapsto \Delta_{n-1}$

Notes on PHP_n^{n+1} and relation of tree-like and dag-like proofs

- Using the cut-rule, there are polynomial-size proofs for PHP_n^{n+1} in Gentzen's LK. Without the cut-rule, every proof for PHP_n^{n+1} is at least $\mathcal{O}(2^{n^\delta})$, where $0 < \delta < 1/5^4$
  - Proof is based on polynomial equivalence of Frege systems and LK with cuts (to be introduced later in the course) and to another theorem to be introduced later in the course
- Proof is based on properties of tree-like and dag-like Frege proof system, and the p-simulation relation between Frege proof system and LK.

Conclusions

- We proved the Completeness of LK
- We defined the subformula property for a proof
- We studied effects of the inference rule cut to the length of the minimum proofs
- We mentioned relations between dag-like and tree-like proofs