

Some Afterthoughts on Hopfield Networks

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Abstract. The present paper investigates four relatively independent issues, each in one section, which complete our knowledge regarding the computational aspects of popular Hopfield nets [9]. In Section 2, the computational equivalence of convergent asymmetric and Hopfield nets is shown with respect to the network size. In Section 3, the convergence time of Hopfield nets is analyzed in terms of bit representations. In Section 4, a polynomial time approximate algorithm for the minimum energy problem is shown. In Section 5, the Turing universality of analog Hopfield nets is studied.

1 Introduction

In his 1982 paper [12], John Hopfield introduced a very influential associative memory model which has since come to be known as the discrete-time Hopfield (or symmetric) network. Particularly, Hopfield nets compared with general asymmetric networks have favorable convergence properties. Part of the appeal of Hopfield nets also stems from their natural hardware implementation, e.g., Ising spin glasses [3], optical computers [7], etc. Hopfield nets are well suited for applications that require the capability to remove noise from large binary patterns. Besides associative memory, the proposed uses of Hopfield networks include, e.g., fast approximate solution of combinatorial optimization problems [13, 31]. Although the practical applicability of Hopfield nets seems to be limited because of their low storage capacity, this fundamental model inspired other important neural network architectures such as BAM, Boltzmann machines, etc. [23]. Thus the theoretical analysis of Hopfield nets is also worthy for understanding the computational capabilities of the corresponding models.

We will first briefly specify the model of a finite *discrete recurrent neural network*. The network consists of n simple computational *units* or *neurons*, indexed as $1, \dots, n$, which are connected into a generally cyclic oriented graph or *architecture* in which each edge (i, j) leading from neuron i to j is labelled with an integer *weight* $w(i, j) = w_{ji}$. The absence of a connection within the architecture corresponds to a zero weight between the respective neurons. Special attention will be paid to *Hopfield (symmetric) networks*, whose architecture is an undirected graph with symmetric weights $w(i, j) = w(j, i)$ for every i, j .

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We will mostly consider the *synchronous* computational dynamics of the network, working in *fully parallel mode*, which determines the evolution of the *network state* $\mathbf{y}^{(t)} = (y_1^{(t)}, \dots, y_n^{(t)}) \in \{0, 1\}^n$ for all discrete time instants $t = 0, 1, \dots$ as follows. At the beginning of the computation, the network is placed in an *initial state* $\mathbf{y}^{(0)}$ which may include an external input. At discrete time $t \geq 0$, each neuron $j = 1, \dots, n$ collects its binary *inputs* from the *states (outputs)* $y_i^{(t)} \in \{0, 1\}$ of incident neurons i . Then its integer *excitation* $\xi_j^{(t)} = \sum_{i=0}^n w_{ji} y_i^{(t)}$ ($j = 1, \dots, n$) is computed as the respective weighted sum of inputs including an integer *bias* w_{j0} which can be viewed as the weight of the formal constant unit input $y_0^{(t)} = 1$. At the next instant $t + 1$, an *activation function* σ is applied to $\xi_j^{(t)}$ for all neurons $j = 1, \dots, n$ in order to determine the new network state $\mathbf{y}^{(t+1)}$ as follows:

$$y_j^{(t+1)} = \sigma \left(\xi_j^{(t)} \right) \quad j = 1, \dots, n \quad (1)$$

where a *binary-state* neural network employs the *hard limiter (or threshold)* activation function

$$\sigma(\xi) = \begin{cases} 1 & \text{for } \xi \geq 0 \\ 0 & \text{for } \xi < 0. \end{cases} \quad (2)$$

Alternative computational dynamics are also possible in Hopfield nets. For example, under *sequential mode* only one neuron updates its state according to (1) at each time instant while the remaining neurons do not change their outputs. Or in Section 5 we will deal with the finite *analog-state* discrete-time recurrent neural networks which, instead of the threshold activation function (2), employ e.g. the *saturated-linear* sigmoid activation function

$$\sigma(\xi) = \begin{cases} 1 & \text{for } \xi > 1 \\ \xi & \text{for } 0 \leq \xi \leq 1 \\ 0 & \text{for } \xi < 0. \end{cases} \quad (3)$$

Hence the states of analog neurons are real numbers within the interval $[0, 1]$, and similarly the weights (including biases) are allowed to be reals.

The fundamental property of the symmetric net is that a bounded Liapunov, or ‘energy’ function can be defined on the state space which is properly decreasing along any nonconstant computation path (*productive* computation). Namely, for a sequential computation of a Hopfield net (for the simplicity, with zero feedbacks $w_{jj} = 0$ and biases $w_{j0} = 0$, and non-zero excitations $\xi_j^{(t)} \neq 0$, $j = 1, \dots, n$) an *energy* associated with state $\mathbf{y}^{(t)}$ at time $t \geq 0$ can be defined as follows:

$$E \left(\mathbf{y}^{(t)} \right) = E(t) = -\frac{1}{2} \sum_{j=1}^n \sum_{i=1}^n w_{ji} y_i^{(t)} y_j^{(t)} \quad (4)$$

for which Hopfield showed that $E(t) \leq E(t-1) - 1$ for every $t \geq 1$ of a productive computation [12]. Moreover, the energy function (4) is bounded, i.e. $|E(t)| \leq W$ where

$$W = \frac{1}{2} \sum_{j=1}^n \sum_{i=0}^n |w_{ji}| \quad (5)$$

is called the *weight* of the network. Hence, the computation must converge to a stable state within time $O(W)$. An analogous result can be shown for parallel update where a cycle of length at most two different states may appear [22].

The present paper discusses four relatively independent issues regarding the computational properties of Hopfield networks and presents the corresponding new results and observations concerning the computational equivalence of asymmetric and Hopfield networks, convergence time analysis, polynomial time approximate solution of the minimum energy problem, and the Turing universality of analog Hopfield nets. Unfortunately, the proofs here are sketched or omitted due to the lack of space and can be found in the respective draft version [29].

2 A Size-Optimal Simulation of Asymmetric Networks

The computational power of Hopfield nets is properly less than that of asymmetric networks because of their different asymptotic behavior. Hopfield nets cannot enter the limit cycle of a given length as the asymmetric networks can. However, it is known [20] that this is the *only* feature that cannot be reproduced, in the sense that any *converging* fully parallel computation by a network of n discrete-time binary neurons, with in general asymmetric interconnections, can be simulated by a Hopfield net of quadratic size $O(n^2)$. More precisely, there exists a subset of neurons in the respective Hopfield net whose states correspond to the original convergent asymmetric computation in the course of simulation possibly with some constant time overhead per each original update. The idea behind this simulation is that each directed edge is implemented by a small symmetric subnetwork which receives energy support from a symmetric clock subnetwork (a binary counter) [11] in order to propagate a signal in the right direction. This result may also be interpreted within the context of infinite families of neural networks which, each for one input length, can be exploited for universal computations (similarly as circuit families). Thus the infinite sequences of discrete symmetric networks with polynomial number of neurons in terms of the input length are computationally equivalent to (nonuniform) polynomially space-bounded Turing machines, i.e. they compute the complexity class PSPACE/poly or P/poly when polynomial weights are considered [20].

In the following theorem the construction from [20] is improved by reducing the number of neurons in the simulating symmetric network to the linear size $6n + 2$ which is asymptotically optimal. This is achieved by simulating the neurons (instead of edges) whose states are updated by means of the clock technique. A similar idea was used for an analogous continuous-time simulation [28]. This result can be interpreted in the sense that convergent asymmetric networks are computationally equivalent with symmetric ones to a greater degree when considering also the network size.

Theorem 1. *Any fully parallel computation by a recurrent neural network of n binary neurons, with generally asymmetric weights, which converges within t^* discrete updates can be simulated by a Hopfield net with $6n + 2$ neurons within $4t^*$ discrete-time steps.*

Proof. (Sketch) Observe, first, that any converging computation by an asymmetric network of n binary neurons must terminate within $t^* \leq 2^n$ steps. A basic technique used in our proof is the exploitation of an $(n + 1)$ -bit symmetric clock subnetwork (a binary counter) which, using $3n + 1$ units, produces a sequence of 2^n well-controlled oscillations $(0111)^{2^n}$ before it converges. This sequence of clock pulses generated by the least significant counter unit c_0 is used to drive the rest of the network. The construction of the $(n + 1)$ -bit binary counter is omitted. We only assume that the corresponding weights are accommodated so that the clock is not influenced by the simulating subnetwork. In addition, neuron \bar{c}_0 is added which computes the negation of c_0 output.

Then for each neuron j from the asymmetric network, 3 units p_j, q_j, r_j are introduced in the Hopfield net so that p_j represents the new (current) state $y_j^{(t)}$ of j at time $t \geq 1$ while q_j stores the old state $y_j^{(t-1)}$ of j from the preceding time instant $t - 1$, and r_j is an auxiliary neuron realizing the update of the old state. The corresponding symmetric subnetwork simulating one neuron j is depicted in Figure 1 where the parameter W is the network weight (5). Here the symmetric connections between neurons are labelled with corresponding weights, and the biases are indicated by the edges drawn without an originating unit. In the sequel the symmetric weights in the Hopfield net will be denoted by w' whereas w denotes the original asymmetric weights. The total number of units simulating the asymmetric network is $3n + 1$ (including \bar{c}_0) which, together with the clock size $3n + 1$, gives the desired $6n + 2$ neurons of the Hopfield net.

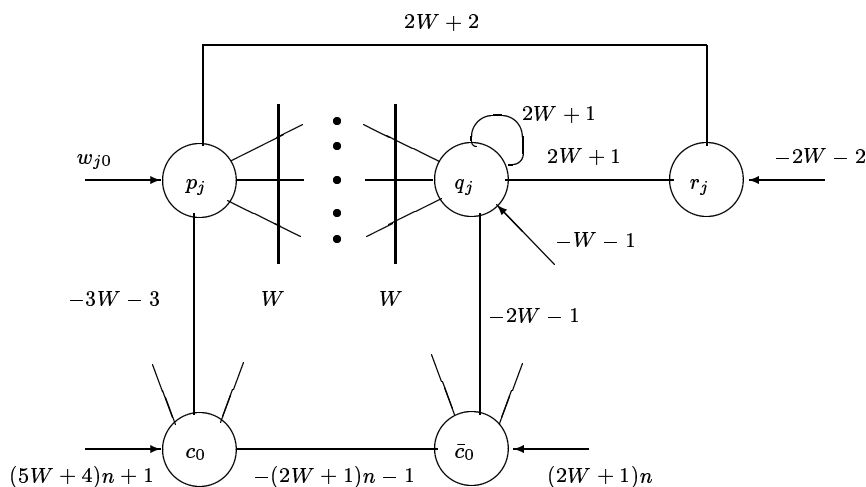


Fig. 1. Symmetric simulation of neuron j

At the beginning of the simulation all the neurons in the Hopfield net are initially passive (their states are zero) except for those units q_j corresponding to

the original initially active neurons j , i.e. $y_j^{(0)} = 1$. Then an asymmetric network update at time $t \geq 1$ is simulated by a cycle of four steps in the Hopfield net as follows. In the first step, unit c_0 fires and remains active until its state is changed by the clock since its large positive bias makes it independent on all the n neurons p_j . Also the unit \bar{c}_0 fires because it computes the negation of c_0 that was initially passive. At the same time each neuron p_j computes the new state $y_j^{(t)}$ from the old ones $y_i^{(t-1)}$ which are stored in corresponding units q_i . Thus each neuron p_j is connected with units q_i via the original weights $w'(q_i, p_j) = w(i, j)$ and also its bias $w'(0, p_j) = w(0, j)$ is preserved. So far, unit q_j keeps the old state $y_j^{(t-1)}$ due to its feedback. In the second step, the new state $y_j^{(t)}$ is copied from p_j to r_j , and the active neuron c_0 makes each neuron p_j passive by means of a large negative weight which exceeds the positive influence from units q_i ($i = 1, \dots, n$) including its bias $w(0, p_j)$ according to (5). Similarly, the active neuron \bar{c}_0 erases the old state $y_j^{(t-1)}$ from each neuron q_j by making it passive with the help of a large negative weight which exceeds its feedback and the positive influence from units p_i ($i = 1, \dots, n$). Finally, also neuron \bar{c}_0 becomes passive since c_0 was active. In the third step, the current state $y_j^{(t)}$ is copied from r_j to q_j since all the remaining incident neurons p_i and \bar{c}_0 are and remain passive due to c_0 being active. Therefore also unit r_j becomes passive. In the fourth step, c_0 becomes passive and the state $y_j^{(t)}$, being called old from now on, is stored in q_j . Thus the Hopfield net finds itself at the starting point of the next asymmetric network update simulation at time $t+1$ which proceeds in the same way. Hence the whole simulation is achieved within $4t^*$ discrete-time steps. \square

3 Convergence Time Analysis

In this section the *convergence time* in Hopfield networks, which is the number of discrete updates until the network converges, will be analyzed. We will consider only the worst case bounds while the average-case analysis can be found in [17]. Obviously, there are exactly 2^n different states in a network with n binary neurons which yields trivial 2^n upper bound on the convergence time in symmetric networks of size n . On the other hand, the symmetric clock network [11] which is used in the proof of Theorem 1 represents an explicit example of a Hopfield net whose convergence time is exponential with respect to n . Namely, this gives $\Omega(2^{n/3})$ lower bound on the convergence time of Hopfield nets since the respective $(k+1)$ -bit binary counter requires $n = 3k + 1$ neurons.

However, the above-mentioned bounds do not take the weight size into account. The corresponding upper bound $O(W)$ is derived from the energy function (see Section 1) which can even be made more accurate by using a slightly different energy function [8]. This yields the polynomial upper bound on the convergence time of Hopfield nets with polynomial weights. Similar arguments can be used for fully parallel updates.

In the following theorem these results will be translated into the convergence time bounds with respect to the length of bit representations of Hopfield

nets. Namely, for a symmetric network which is described within M bits, the convergence-time lower and upper bounds, $2^{\Omega(M^{1/3})}$ and $2^{O(M^{1/2})}$, respectively will be observed. It is an open problem whether these upper or lower bounds can be improved. This is an important issue since the convergence-time results for binary-state networks could be compared with those for analog-state (or even continuous-time) networks in which the precision of real weight parameters (i.e. the representation length) plays an important role. For example, there exists an analog-state symmetric network with an encoding size of M bits that converges after $2^{\Omega(g(M))}$ continuous-time units, where $g(M)$ is an arbitrary continuous function such that $g(M) = o(M)$, $g(M) = \Omega(M^{2/3})$, and $M/g(M)$ is increasing [28]. From the result presented here it follows that the computation of this analog symmetric network terminates later than that of any other discrete Hopfield net of the same representation size. This approach also appears to be more rigorous since we express the convergence time with respect to the full descriptive complexity of the Hopfield net instead of to the number of neurons which captures its computational sources only partially.

Theorem 2. *There exists a Hopfield network with an encoding size of M bits that converges after $2^{\Omega(M^{1/3})}$ updates and any computation of a symmetric network with a binary representation of M bits terminates within $2^{O(M^{1/2})}$ discrete computational steps.*

Proof. (Sketch) For the underlying lower bound the clock network from the proof of Theorem 1 can again be exploited. For the upper bound, consider a Hopfield network with an M -bit representation that converges after $T(M)$ updates. A major part of this M -bit representation consists of m binary encodings of weights w_1, \dots, w_m of the corresponding lengths M_1, \dots, M_m where $\sum_{r=1}^m M_r = \Theta(M)$. Clearly, there must be at least $T(M)$ different energy levels corresponding to the states visited during the computation. Thus the underlying weights must produce at least $S \geq T(M)$ different sums $\sum_{r \in A} w_r$ for $A \subseteq \{1, \dots, m\}$ where w_r for $r \in A$ agrees with w_{j_i} for $y_i = y_j = 1$ in (4). So, it is sufficient to upper bound the number of different sums over m weights whose binary representations form a $\Theta(M)$ -bit string altogether. This yields $T(M) \leq 2^{O(M^{1/2})}$. \square

4 Approximating the Minimum Energy Problem

Another important issue in Hopfield nets is the *MIN ENERGY* or *GROUND STATE problem* of finding a network state with minimal energy (4) for a given symmetric neural network. Remember that in (4) it is assumed, for reasons of simplicity, that $w_{jj} = 0$ and $w_{j0} = 0$ for $j = 1, \dots, n$. In addition, without loss of generality [21], we will work throughout this section with frequently used bipolar states $-1, 1$ of neurons instead of binary ones $0, 1$ introduced in (2) where 0 is now replaced by -1 . This problem appears to be of a special interest since many hard combinatorial optimization problems have been heuristically solved by minimizing the energy in Hopfield nets [1, 13]. This issue is also important in

statistical physics which originally inspired the Hopfield net models, e.g. Ising spin glasses [3].

Unfortunately, the decision version of the MIN ENERGY problem, i.e. whether there exists a network state having an energy less than the prescribed value, is NP-complete. This can be observed from the above-mentioned reductions of hard optimization problems to MIN ENERGY. For an explicit NP-completeness proof see e.g. [32] where a reduction from SAT is exploited. On the other hand there is a MIN ENERGY polynomial algorithm for special cases of Hopfield nets whose architectures are planar lattices [6] or planar graphs [3].

Perhaps, the most direct and frequently used reduction to MIN ENERGY is from the MAX CUT problem (see e.g. [4]) which, given an undirected graph $G = (V, E)$ with an integer edge evaluation $c : E \rightarrow \mathcal{Z}$, is the issue of finding a cut $V_1 \subseteq V$ which maximizes the cut size

$$c(V_1) = \sum_{\{i,j\} \in E, i \in V_1, j \notin V_1} c(\{i,j\}) - \sum_{\{i,j\} \in E, c(\{i,j\}) < 0} c(\{i,j\}). \quad (6)$$

In fact, this is a generalized version of MAX CUT that allows negative edge evaluations necessary for the opposite reduction from MIN ENERGY to MAX CUT. Recently, a new randomized approximation algorithm with a high performance guarantee $\alpha = 0.87856$ for this MAX CUT formulation has been proposed [10] and later derandomized [19] which we will exploit for approximating the MIN ENERGY problem. Namely, we will observe that MIN ENERGY can be approximated in a polynomial time within the absolute error less than $0.243W$ where W is the network weight (5). For $W = O(n^2)$ which is satisfied by e.g. Hopfield nets with n neurons and constant weights, this result matches the lower bound $\Omega(n^{2-\varepsilon})$ which cannot be guaranteed by any approximate polynomial time MIN ENERGY algorithm for every $\varepsilon > 0$ [4], unless $P = NP$. In addition, an approximate polynomial time MIN ENERGY algorithm with absolute error $O(n/\log n)$ is also known in a special case of Hopfield nets whose architectures are two-level grids [5].

Theorem 3. *The MIN ENERGY problem for Hopfield nets can be approximated in a polynomial time within the absolute error less than $0.243W$ where W is the network weight (5).*

Proof. (Sketch) We will first recall the well-known simple reduction between MIN ENERGY and MAX CUT problems. For a Hopfield network with architecture G and weights $w(i, j)$ we can easily define the corresponding instance $G = (V, E); c$ of MAX CUT with the edge evaluation $c(\{i, j\}) = -w(i, j)$ for $\{i, j\} \in E$. It can easily be shown that any cut $V_1 \subseteq V$ of G corresponds to a Hopfield net state $\mathbf{y} \in \{-1, 1\}^n$ where $y_i = 1$ if $i \in V_1$ and $y_i = -1$ for $j \in V \setminus V_1$, so that the respective cut size $c(V_1)$ is related to the underlying energy $E(\mathbf{y}) = W - 2c(V_1)$. This implies that the minimum energy state corresponds to the maximum cut.

Now, the approximate polynomial time algorithm from [10] can be employed to solve instance $G = (V, E); c$ of the MAX CUT problem which provides a cut V_1 whose size $c(V_1) \geq \alpha c^*$ is guaranteed to be at least $\alpha = 0.87856$ times the

maximum cut size c^* . Let cut V_1 correspond to the Hopfield network state \mathbf{y} which implies $c(V_1) = 1/2(W - E(\mathbf{y}))$. Hence, we get a guarantee $W - E(\mathbf{y}) \geq \alpha(W - E^*)$ where E^* is the minimum energy corresponding to the maximum cut c^* which leads to $E(\mathbf{y}) - E^* \leq (1 - \alpha)(W - E^*)$. Since $|E^*| \leq W$, we obtain the desired guarantee for the absolute error $E(\mathbf{y}) - E^* \leq (1 - \alpha)2W < 0.243W$. \square

5 Turing Universality of Finite Analog Hopfield Nets

In this section we will deal with the computational power of finite *analog-state* discrete-time recurrent neural networks. For the asymmetric analog networks, the computational power is known to increase with the Kolmogorov complexity of real weights [2]. With integer weights such networks are equivalent to finite automata [14, 15, 30], while with rational weights arbitrary Turing machines can be simulated [15, 25]. With arbitrary real weights the network can even have ‘super-Turing’ computational capabilities, e.g. polynomial time computations correspond to the complexity class P/poly and all languages can be recognized within exponential time [24]. On the other hand, any amount of analog noise reduces the computational power of this model to that of finite automata [18].

For finite symmetric networks, only the computational power of binary-state Hopfield nets is fully characterized. Namely, they recognize the so-called Hopfield languages [26] which establish a proper subclass of regular languages and hence, they are less powerful than finite automata. Hopfield languages can also be faithfully recognized by analog symmetric neural networks [18, 27] and this provides the lower bound on their computational power. A natural question arises whether the finite analog Hopfield nets are Turing universal, i.e. whether a Turing machine simulation can be achieved with rational weights similarly as in the asymmetric case [15, 25]. The main problem is that under fully parallel update any analog Hopfield net with rational weights converges to a limit cycle of length at most two [16]. Thus the only possibility of simulating Turing machines is to exploit a sequence of rational network states converging to this limit cycle which seems to be tricky if possible at all. A more reasonable approach is to supply an external clock that produces an infinite sequence of binary pulses providing the symmetric network with an energy support, e.g. for simulating an asymmetric analog network similarly as in Theorem 1. In this way the computational power of the analog Hopfield nets with an external clock is proved to be the same as that of the asymmetric analog networks. Especially for rational weights, this implies that they are Turing universal. The following theorem also completely characterizes the infinite binary sequences by the external clock, which prevent the Hopfield network from converging.

Theorem 4. *Let N be an analog-state recurrent neural network with real asymmetric weights and n neurons working in a fully parallel mode. Then there exists an analog Hopfield net N' with the same maximum Kolmogorov complexity of a real weight as that in N and with $3n + 8$ units such that N' simulates any computation of N for any binary sequence which is generated by an additional external*

input of N^l satisfying the following property. Namely, this sequence must contain the infinite number of substrings of the form $b\bar{x}\bar{b} \in \{0, 1\}^3$ where $b \neq \bar{b}$. In addition, this property is necessary to prevent N^l from converging.

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