Computing with Continuous-Time Liapunov Systems

Jiří Šíma*
ICS, Academy of Sciences of the Czech Rep.,
P.O. Box 5, 182 07 Prague 8, Czech Republic,
& Institute for Theoretical Computer Science
(ITI), Charles University, Prague
sima@cs.cas.cz

Pekka Orponen
Department of Mathematics,
University of Jyväskylä,
P.O. Box 35, FIN-40351 Jyväskylä, Finland,
orponen@math.jyu.fi

ABSTRACT

We establish a fundamental result in the theory of computation by continuous-time dynamical systems, by showing that systems corresponding to so called continuous-time symmetric Hopfield nets are capable of general computation. More precisely, we prove that any function computed by a discrete-time asymmetric recurrent network of n threshold gates can also be computed by a continuous-time symmetrically-coupled Hopfield system of dimension $18n + 7$. Moreover, if the threshold logic network has maximum weight $w_{\max}$ and converges in discrete time t^*, then the corresponding Hopfield system can be designed to operate in continuous time $\Theta(t^*/\varepsilon)$, for any value $0 < \varepsilon < 0.0025$ such that $w_{\max}2^{3n} \leq \varepsilon 2^{1/\varepsilon}$.

The result appears at first sight counterintuitive, because the dynamics of any symmetric Hopfield system is constrained by a *Liapunov*, or *energy function* defined on its state space. In particular, such a system always converges from any initial state towards some stable equilibrium state, and hence cannot exhibit nondamping oscillations, i.e. strictly speaking cannot simulate even a single alternating bit. However, we show that if one only considers *terminating* computations, then the Liapunov constraint can be overcome, and one can in fact embed arbitrarily complicated computations in the dynamics of Liapunov systems with only a modest cost in the system's dimensionality.

In terms of standard discrete computation models, our result implies that any polynomially space-bounded Turing machine can be simulated by a family of polynomial-size continuous-time symmetric Hopfield nets.

1. INTRODUCTION

In recent years, a number of studies have sought to understand the computational characteristics of "natural" dynamical systems. The results achieved include, e.g. universal computation results for several types of ODE's [1, 4], PDE's [20], and discrete iterations [15, 18, 27]. One much studied class of systems are those that are defined by various *neural network* models [7, 9, 25]. This interest is motivated partly by the quest to understand the fundamental limits and possibilities of practical neurocomputing, and partly by the realization that despite their formal simplicity, neural networks are computationally quite powerful, and thus may serve as a useful reference model for investigating more complicated systems. In general, the computational capabilities of discrete-time systems are by now fairly well understood, but in the area of continuous-time systems much work remains to be done (for surveys of this area, see e.g. [19, 23]).

In this paper, we prove a fundamental result concerning the computational power of a class of dynamical systems popularized by John Hopfield in 1984 [11], and known as the "continuous-time symmetric Hopfield nets". (The dynamics of this model were actually already analyzed earlier by Cohen and Grossberg in a more general setting [6], but because of the affinity to the very influential discrete-time binary-state version of the model [10], this additive special case of the Cohen-Grossberg equations has become associated to Hopfield's name.) As practical neural networks, proposed uses of Hopfield-type systems include associative memory [11] and fast approximate solution of combinatorial optimization problems [12], and designs exist for implementing them in analog electrical [11] and optical [28] hardware.

It is well known [6, 11] that the dynamics of any Hopfield-type system with a *symmetric* interconnection matrix is governed by a *Liapunov*, or *energy function*. This is a bounded function defined on the state space of a system, whose values are properly decreasing along any nonconstant trajectory of the system's dynamics. At first sight, the existence of a Liapunov function would appear to severely limit the capabilities of a dynamical system for general computation, because it implies that any trajectory will eventually converge towards some stable equilibrium state. Thus e.g. nondamping oscillations, which seem to be an essential prerequisite of general computation, cannot be created.

Nevertheless, we shall show that infinite oscillations are the *only* feature of general-purpose digital computation that cannot be reproduced in continuous-time symmetric Hopfield systems. More precisely, we prove that any *converging* discrete-time computation described by a recurrent network of n threshold gates (with in general *asymmetric* interconnections) can be embedded in a continuous-time Hopfield

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system of $18n + 7$ variables with a symmetric interconnection weight matrix, and using a saturated-linear "activation function". An experimental validation of this result appeared previously in the extended abstract [30].

A key observation is that any terminating computation by a discrete-time deterministic system with state space $\{0,1\}^n$ must converge within 2^n steps. A basic technique used in our proof is then the construction of a symmetric continuous-time *clock* subsystem—a simulated $(n+2)$-bit binary counter—that, using $8n+7$ variables, produces a sequence of 2^n well-controlled oscillations (generated by the simulated second least significant counter bit) before it converges. This sequence of clock pulses is used to drive the rest of the system where each discrete threshold gate is simulated by a symmetrically coupled subsystem of 10 continuous-time variables. The continuous-time clock is already by itself of some interest from a dynamical systems perspective, because it provides to our knowledge the first known example of a continuous-time Liapunov system whose convergence time grows exponentially in the system dimension [31].

A similar, but considerably simpler, construction was used in the discrete-time setting in [21] to prove the computational equivalence of symmetrically and asymmetrically interconnected convergent threshold logic networks. The original idea for the discrete-time clock network used in [21] stems from [8]. Another related work [22] concerns the simulation of discrete-time threshold logic networks by continuous-time non-Liapunov Hopfield-type systems with *asymmetric* interconnection matrices.

It is quite easy to see [16, 21] that families of polynomial-size threshold networks are computationally equivalent to (nonuniform) polynomially space-bounded Turing machines (more precisely, they compute the complexity class PSPACE/poly [2, p. 105]). By the result in the present paper, we now know that continuous-time symmetric Hopfield systems are also at least as powerful, i.e. given any polynomially space-bounded Turing machine, we can construct a family of polynomial-size continuous-time symmetric Hopfield nets for simulating it. It remains an open question whether this is also an upper bound on the power of such systems.

A related line of study concerns the computational capabilities of finite *discrete-time analog-state* neural networks [25]. Here it is known that the computational power of asymmetric networks using the saturated-linear activation function increases with the Kolmogorov complexity of the weight parameters [3]. With integer weights such networks coincide with threshold logic networks, and so are equivalent to finite automata [13, 14, 33], while with rational weights arbitrary Turing machines can be simulated [14, 27]. With arbitrary real weights the networks can even have "super-Turing" computational capabilities [26]. To some extent similar results also apply to discrete-time *symmetric* analog networks that are computationally equivalent to their asymmetric counterparts when an external clock pulse sequence is provided [32]. On the other hand, it is known that any amount of analog noise reduces the computational power of discrete-time analog systems to that of finite automata [5, 17].

The present paper is organized as follows. After a brief review of the basic definitions in Section 2, our main construction of the symmetric continuous-time Hopfield system simulating a given discrete-time threshold logic network is outlined in Section 3 where its dynamics is also informally explained. The formal verification of this construction, which has the form of a rather tedious case analysis, is given in Section 4. In Section 5 a numerical simulation example witnessing the validity of the construction is presented. Section 6 concludes with some open problems.

2. PRELIMINARIES

A *threshold logic network (TLN)* consists of n weighted threshold gates, indexed as $1,\ldots,n$, that are located in the nodes of a generally cyclic directed graph. Each edge (i,j) in this graph, leading from gate i to gate j, is labeled with an integer *interconnection weight* $w(i,j) = w_{ji} \in \mathbf{Z}$. The absence of an edge in the graph indicates a zero weight between the respective gates, and vice versa.

An instantaneous *state* of a TLN is described by a vector $\mathbf{y} = (y_1,\ldots,y_n) \in \{0,1\}^n$ composed of binary *outputs* (*states*) y_j from particular threshold gates $j = 1,\ldots,n$. The state evolution in time is defined as follows. (We only consider so called "fully parallel" dynamics of networks here.) Initially the TLN is placed in some initial state $\mathbf{y}^{(0)}$, which may include an external input. Given a state $\mathbf{y}^{(t)}$ of the network at a discrete time instant $t = 0,1,\ldots$, its state $\mathbf{y}^{(t+1)}$ at next time $t+1$ is computed componentwise as

$$y_j^{(t+1)} = H\left(\xi_j^{(t)}\right), \quad j = 1,\ldots,n, \tag{1}$$

where

$$\xi_j^{(t)} = \sum_{i=0}^{n} w_{ji} y_i^{(t)} \tag{2}$$

is the integer *excitation* of gate j at time t, and H is the *Heaviside* activation function:

$$H(\xi) = \begin{cases} 1 & \text{for } \xi \geq 0 \\ 0 & \text{for } \xi < 0 . \end{cases} \tag{3}$$

The excitation value (2) of each gate j includes an integer local *bias* term $w_{j0} \in \mathbf{Z}$, which is formally modeled as a connection weight from a constant unit-output gate $y_0^{(t)} \equiv 1$. We denote by

$$w_{\max} = \max_{j=1,\ldots,n;\, i=0,\ldots,n} |w_{ji}| \tag{4}$$

the *maximum weight* size in the network.

A *Hopfield system* of dimension m is defined by a set of m symmetrically coupled ordinary differential equations in real variables $y_1,\ldots,y_m \in [0,1]$:

$$\frac{dy_p}{dt}(t) = -y_p(t) + \sigma(\xi_p(t)), \quad p = 1,\ldots,m, \tag{5}$$

where

$$\xi_p(t) = \sum_{q=0}^{m} v(q,p) y_q(t) \tag{6}$$

is the real-valued excitation for site $p = 1,\ldots,m$, the real values $v(q,p) = v(p,q)$ for all $1 \leq p,q \leq m$ are the weights forming a symmetrical coupling matrix, $v(0,p)$ is the real bias associated with each $1 \leq p \leq m$, and σ is some nonlinear "sigmoidal" activation function. We fix the activation function σ to be the *saturated linear* map:

$$\sigma(\xi) = \begin{cases} 1 & \text{for } \xi \geq 1 \\ \xi & \text{for } 0 < \xi < 1 \\ 0 & \text{for } \xi \leq 0 . \end{cases} \tag{7}$$

A convenient way of representing such a system of type (5), which we shall adopt, is to interpret each of the variables y_p as the real-valued state (output) of a computational unit p evolving in continuous time, and to represent the symmetric coupling weight $v(p,q)$ as the weight on an undirected edge connecting unit p to unit q. Such a *continuous-time Hopfield net* can be used for computation analogously to a TLN, with the initial system state $\mathbf{y}(0) \in [0,1]^m$ encoding the initial conditions, including any possible external input.

The dynamics of a continuous-time *symmetric* Hopfield system of type (5) is controlled by the following *Liapunov* or *energy* function, introduced in [6, 11]:

$$E(\mathbf{y}) = -\frac{1}{2}\sum_{p=1}^{m}\sum_{q=1}^{m} v(q,p)y_q y_p - \sum_{p=1}^{m} v(0,p)y_p + \sum_{p=1}^{m}\int_0^{y_p} \sigma^{-1}(y)dy\,. \tag{8}$$

The characteristic properties of function $E(\mathbf{y})$ are that it is bounded on the system's state space $[0,1]^m$, and that it is properly decreasing (i.e. $dE/dt < 0$) along any nonconstant trajectory of the system's dynamics. It then follows that the system (5) always converges, from any initial condition, towards some stable equilibrium state with $dy_p/dt = 0$ for all $p = 1, \ldots, m$.

3. THE SIMULATION

We shall now show how to simulate the computations of a given TLN $\mathcal{N}$ on a somewhat bigger Hopfield system $\mathcal{H}$. In our simulation, the binary states of the gates in $\mathcal{N}$ will be represented by excitations (6) of the corresponding analog units in $\mathcal{H}$ that are either above the upper saturation threshold of 1 or below the lower saturation threshold of 0 for the activation function σ. For brevity, we shall simply say that a unit p is *saturated* at 0 or 1 at time t if its excitation satisfies $\xi_p(t) \le 0$ or $\xi_p(t) \ge 1$, respectively. Correspondingly, we say that p is *unsaturated* when $0 < \xi_p(t) < 1$ (see (7)). Note that we use the *excitations* of continuous-time units in $\mathcal{H}$ rather than their actual *states* to represent binary values. The following theorem summarizes the result:

THEOREM 1. *Any computation by a threshold logic network $\mathcal{N}$ of n gates with (asymmetric) weights of maximum size $w_{\max}$, converging within t^* discrete update steps, can be simulated by a continuous-time symmetric Hopfield system $\mathcal{H}$ of dimension $m = 18n + 7$, within continuous time $\Theta(t^*/\varepsilon)$ for any positive real $0 < \varepsilon < 0.0025$ such that $w_{\max} 2^{3n} \le \varepsilon 2^{1/\varepsilon}$.*

PROOF. (Sketch.) Since the TLN $\mathcal{N}$ defines a deterministic dynamical system over the state space $\{0,1\}^n$, any converging computation by $\mathcal{N}$ must terminate within $t^* \le 2^n$ discrete steps. To obtain the simulating Hopfield system $\mathcal{H}$ we first construct, in a manner presented in Figures 1, 2, an $(8n+7)$-variable Hopfield *clock* subsystem $\mathcal{C} = \mathcal{C}_{n+1}$ that simulates a discrete $(n+2)$-bit binary counter. When the system $\mathcal{C}$ is initialized in the zero initial state, its *interface* unit x_1 will exhibit a sequence of 2^n well-controlled oscillations before $\mathcal{C}$ converges. Corresponding to each of the gates $j = 1, \ldots, n$ in $\mathcal{N}$, we further construct a 10-variable Hopfield *gate* subsystem $\mathcal{G}_j$ as shown in Figure 3. Each "clock pulse" from interface unit x_1 of $\mathcal{C}$ is then exploited to "drive" a simulation of one parallel discrete update iteration by the gate systems $\mathcal{G}_j$ ($j = 1, \ldots, n$). We shall now first discuss this construction intuitively while its correctness will formally be verified later in Section 4.

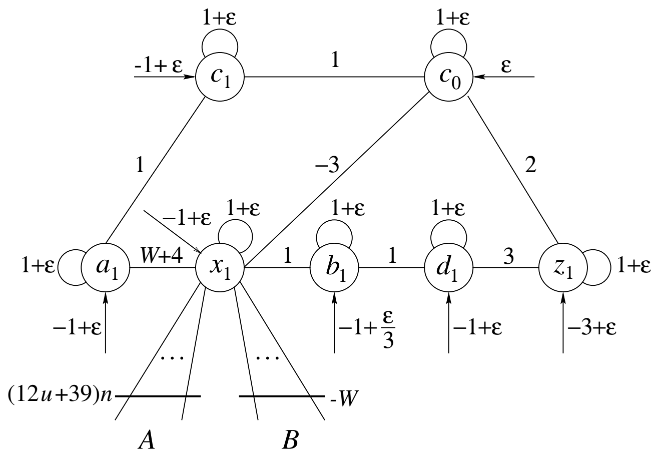


Figure 1: A 2-bit continuous-time counter $\mathcal{C}_1$.

The construction of the counter system $\mathcal{C}_{n+1}$ of "order (n+1)" will be described by induction on n, starting with the corresponding 2-bit counter network $\mathcal{C}_1$ presented in Figure 1. (The edges in this graph drawn without an originating unit correspond to the bias terms.) Assume that initially all the variables, i.e. states of the units indicated in this network are zero. Now because of the positive bias term $v(0,c_0) = \varepsilon > 0$, the state of the least significant counter unit c_0 of "order 0" has a positive initial excitation. Thus, the feedback coupling $v(c_0,c_0) = 1+\varepsilon$ causes the state value of c_0 to gradually grow towards 1. Eventually c_0 saturates at 1, at which point we say that the unit c_0 becomes *active* or *fires*. (Recall that we associate the simulated discrete behavior to the excitations of the units rather than their outputs. The external state of c_0 of course evolves continuously, and exhibits no abrupt "firing" transitions.) Thus, c_0 simulates the counting from 0 to 1 as required. This trick of gradual transition from 0 to 1, formally described in Lemma 3 below, is used repeatedly throughout our construction of $\mathcal{C}$.

The remaining six units in $\mathcal{C}_1$ are of "order 1". They function similarly to the corresponding units of higher orders, so we will describe only the inductive construction for general order $k > 1$. The eventual interconnection of the clock network $\mathcal{C}$ to the gate subnetworks $\mathcal{G}_j$ ($j = 1, \ldots, n$) is taken into account in the weight $v(a_1, x_1) = W + 4$, which includes a large positive parameter

$$W = \frac{3}{2}\left(nu + 28\left(n + \sum_{j=1}^{n}|w_{j0}|\right)\right) > 0\,, \tag{9}$$

where

$$u = 28 \max_{j=1,\ldots,n}\left(-\sum_{i=1;\, w_{ji}\le 0}^{n} w_{ji},\ \sum_{i=1;\, w_{ji}\ge 0}^{n} w_{ji}\right) \ge 0\,, \tag{10}$$

and w_{ji} are the weights of the original TLN $\mathcal{N}$ to be simulated. This large weight is needed because the interface unit x_1 transfers the pulses generated by the clock $\mathcal{C}$ to all of the gate subnetworks $\mathcal{G}_j$, and its operation within $\mathcal{C}$ must not be affected by feedback effects from the $\mathcal{G}_j$.

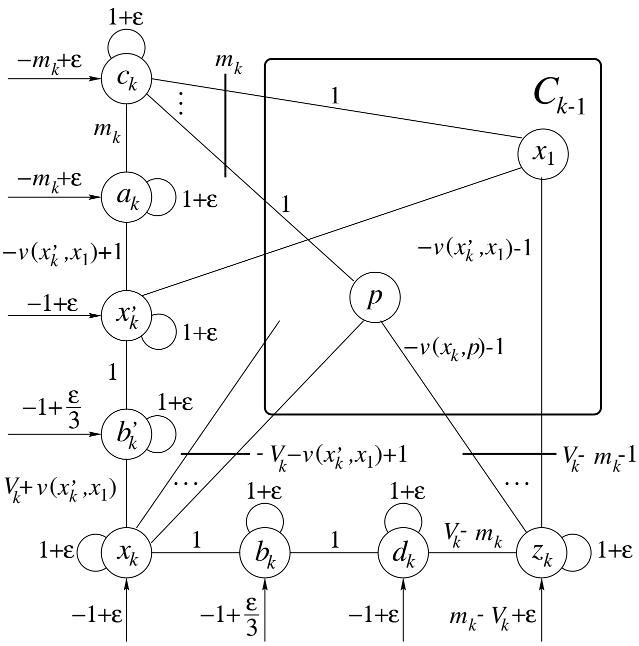


Figure 2: Inductive construction of $\mathcal{C}_k$.

For the induction step depicted in Figure 2, assume that an "order $(k-1)$" counter network $\mathcal{C}_{k-1}$ ($1 < k \le n+1$) has been constructed, containing the k counter units $c_0, \ldots, c_{k-1}$, together with auxiliary units $a_\ell, x_\ell, b_\ell, d_\ell, z_\ell$ ($\ell = 1, \ldots, k-1$), and for $k > 2$ also x'_ℓ, b'_ℓ ($\ell = 2, \ldots, k-1$) for a total of $m_k = 8k - 9$ units. Then the next counter unit c_k is connected to all the m_k units $p \in \mathcal{C}_{k-1}$ via unit weights which, together with its bias, make c_k to fire shortly after all these units are active, i.e. when the simulated counting from 0 to $2^k - 1$ has been accomplished. In addition, unit c_k is connected to a sequence of seven auxiliary units $a_k, x'_k, b'_k, x_k, b_k, d_k, z_k$, which are being, one by one, activated after c_k fires (Lemma 3).

The purpose of the auxiliary units a_k, b'_k, b_k, d_k is only to slow down the continuous-time state flow. The units x'_k, x_k, on the other hand, are used to reset all the lower-order units in $\mathcal{C}_{k-1}$ back to values near 0 after c_k fires (Lemma 2.2b). Units in $\mathcal{C}_{k-1}$ that saturate at 0 are then called *passive*. To achieve this effect, x'_k is connected to x_1 via a large negative weight $v(x'_k, x_1) = -S_{kx_1} - (12u+39)n$, and x_k is linked with each $p \in \mathcal{C}_{k-1} \setminus \{x_1\}$ via negative weight $v(x_k, p) = -S_{kp}$. The value

$$S_{kp} = \left\lceil v(c_k, p) + \sum_{q \in \mathcal{C}_{k-1};\, v(q,p) > 0} v(q, p) \right\rceil, \quad p \in \mathcal{C}_{k-1} \quad (11)$$

is chosen so that it exceeds the mutual positive influence of units in $\mathcal{C}_{k-1} \cup \{c_k\}$, and term $-(12u + 39)n$ balances the total positive influence on x_1 originating from units A in the gate subnetworks (see below). The value of parameter

$$V_k = 1 - v(x'_k, x_1) - \sum_{p \in \mathcal{C}_{k-1} \setminus \{x_1\}} v(x_k, p) \qquad (12)$$

is determined so that the state of x_k is independent of the states of $p \in \mathcal{C}_{k-1}$. Note that the interface unit $x_1 \in \mathcal{C}_{k-1}$ is first suppressed separately by x'_k (Lemma 4) while the remaining units in $\mathcal{C}_{k-1} \setminus \{x_1\}$ are active. Only after x_1 becomes passive, the other units are reseted by x_k. Finally, unit z_k balances the negative influence of x'_k, x_k on $\mathcal{C}_{k-1}$ so that the first k counter bits can again count from 0 to $2^k - 1$ but now with c_k being active. This is achieved by exact weights $v(z_k, x_1) = -v(x'_k, x_1) - 1$ and $v(z_k, p) = -v(x_k, p) - 1$ for $p \in \mathcal{C}_{k-1} \setminus \{x_1\}$ in which the -1 compensates for $v(c_k, p) = 1$. Clearly, units $p \in \mathcal{C}_{k-1}$ cannot reversely affect z_k since their maximal contribution $\sum_{p \in \mathcal{C}_{k-1}} v(p, z_k) = -m_k - v(x'_k, x_1) - \sum_{p \in \mathcal{C}_{k-1} \setminus \{x_1\}} v(x_k, p) = V_k - m_k - 1$ to the excitation of z_k cannot overcome its bias. This completes the induction step of the counter network construction.

In the construction of a subnetwork $\mathcal{G}_j$ ($j = 1, \ldots, n$) for simulating threshold gate j (Figure 3), we apply a device used previously in the context of discrete (acyclic) threshold circuit simulation [24]: by arranging the absolute values of weights and biases in a symmetric network in a decreasing sequence, we can force a signal to propagate (Lemma 2.2b) only in the direction of decreasing weights, i.e. from top to bottom in Figure 3. However, to make recurrent computation possible, the "bottom" units ψ_j, ω_j in $\mathcal{G}_j$ are further connected to the "top" units α_j, β_j—but only via small weights that need additional strong support from the clock $\mathcal{C}$ in order to transmit a signal. To each discrete computational step of the simulated TLN $\mathcal{N}$ then corresponds a two-phase continuous-time state dynamics of the networks $\mathcal{G}_j$, controlled by one oscillation of the clock interface unit $x_1 \in \mathcal{C}$. In particular, the next simulated state of $\mathcal{N}$ is computed in the first phase, and this replaces the old state in the second phase.

In the first phase, the interface unit x_1 is passive, first reseting and locking the units from set $A = \{\alpha_j, \beta_j;\, j = 1, \ldots, n\}$ in their state values near 0, but then allowing the states of units in set $B = \{\phi_j, \chi_j;\, j = 1, \ldots, n\}$ to evolve (Lemma 4). The simulated state $y_j^{(t)} \in \{0, 1\}$ of a threshold gate j at discrete time instant t is doubly represented by the states of two continuous-time units $\pi_j, \varrho_j \in \mathcal{G}_j$ that are both saturated at value $y_j^{(t)}$. Thus, the units π_j, ϱ_j are either both passive due to their negative biases, if $y_j^{(t)} = 0$, or they are supporting each other in the active state via their positive interconnection weight, if $y_j^{(t)} = 1$. (Consequently, also in setting up the initial state of $\mathcal{H}$, units $\pi_j, \varrho_j \in \mathcal{G}_j$ such that $y_j^{(0)} = 1$ in $\mathcal{N}$ are the only ones that are initialized to 1; all the other units in $\mathcal{H}$ are initialized to 0.) In this phase, the value corresponding to the new state $y_j^{(t+1)}$ of threshold gate j is computed in unit $\phi_j \in \mathcal{G}_j$, based on its connections to the units $\pi_i \in \mathcal{G}_i$ that represent the simulated states $y_i^{(t)}$ of $\mathcal{N}$ at the previous time instant, via symmetric weights $v(\pi_i, \phi_j) = 28w_{ji}$, where w_{ji} are the weights of $\mathcal{N}$. (In fact, these weights are appropriately adjusted so that ϕ_j is not influenced by ψ_j, ω_j while the original function of j is preserved [24].) Similarly, unit χ_j computes the negation of $y_j^{(t+1)}$ from units $\varrho_i \in \mathcal{G}_i$ via the opposite weights. Note that the parameter u defined in (10) ensures that the state evolution of ϕ_j, χ_j cannot reversely affect the dynamics of units π_i, ϱ_i. Also units $\phi_j, \chi_j \in \mathcal{G}_j$ are never simultaneously active, since either they are both passive, or each of them represents the negation of a binary state stored in the other. This implies that the parameter W introduced in (9) and included in the weight $v(a_1, x_1)$ is sufficient for activating x_1 regardless of the total negative influence from units in B.

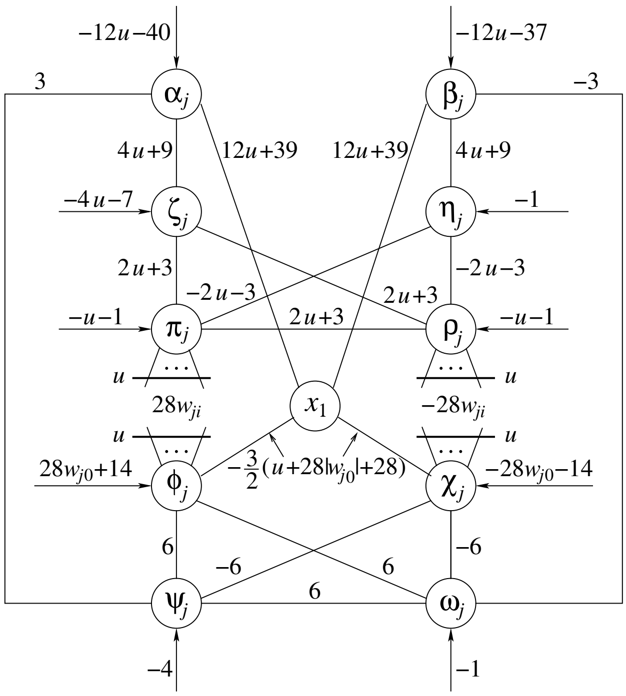


Figure 3: A continuous-time gate-simulation net $\mathcal{G}_j$.

Further, units ϕ_j, χ_j store the new state $y_j^{(t+1)}$ into both ψ_j and ω_j in such a way that either ϕ_j ensures activating ψ_j, ω_j through positive weights if $y_j^{(t+1)} = 1$ or χ_j arranges their reseting by means of negative weights when $y_j^{(t+1)} = 0$. Then, the gate network $\mathcal{G}_j$ becomes temporarily stable until unit x_1 is activated.

In the second phase, active unit x_1 locks the units in B before releasing the units in A (Lemma 4). Unit α_j then receives value $y_j^{(t+1)}$ from unit ψ_j, as the support from x_1 balances the negative bias of α_j so that the small positive weight from ψ_j can have an influence on α_j. Similarly, unit β_j computes the negation of the state $y_j^{(t+1)}$ represented by unit ω_j. State $y_j^{(t+1)}$ and its negation are further propagated in $\mathcal{G}_j$ from α_j, β_j to ζ_j, η_j, respectively, and these units then update the simulated state of threshold gate j stored in π_j, ϱ_j by its new value $y_j^{(t+1)}$. After that, $\mathcal{G}_j$ is temporarily stabilized until x_1 becomes passive again, and computing the new discrete state $y_j^{(t+2)}$ is initiated. Recall that units $\alpha_j, \beta_j \in \mathcal{G}_j$ are never simultaneously active, which implies that the term $-(12u+39)n$ in $v(x'_k, x_1)$ $(k > 1)$ suffices to suppress the total positive influence from A on x_1. This completes the construction of the gate networks $\mathcal{G}_j$.

4. FORMAL VERIFICATION

Now the correct state evolution of the Hopfield system $\mathcal{H}$ described above needs to be verified. This is achieved by a sequence of lemmas analyzing the behavior of the system of differential equations (5). Due to lack of space, the proofs are only sketched here; a full presentation can be found in [29]. Lemma 1 first upper bounds the maximum sum of absolute values of weights incident on any unit in $\mathcal{H}$. Lemma 2 then describes explicitly the continuous-time state evolution for saturated units. An analysis of how the decreasing *defects*, or deviations from limit values in the states of saturated units, affect the excitation of any unit in the continuous-time Hopfield net reveals that its units actually approximate the discrete update rule (1) after a certain transient time, provided that the incident saturated units remain saturated. The proof of Lemma 2 follows from the dynamics equations (5) and Lemma 1. Furthermore, the transfer of the activity in the clock $\mathcal{C}$ from a unit to a subsequent one, when all the incident units are saturated, will be analyzed explicitly and its duration time will be calculated in Lemma 3. (But note that the dynamics of unit c_0 at time $t = 0$ slightly differs from this analysis.) The result is also generalized for the case when some of the incident units may unsaturate. Finally, Lemma 4 will verify that the clock interface unit $x_1 \in \mathcal{C}$ correctly synchronizes the simulation in the gate networks $\mathcal{G}_j$, i.e. locking the gates in A or B always precedes unlocking the gates in B or A, respectively.

LEMMA 1. *For any unit $p \in \mathcal{H}$ in the Hopfield system constructed above, the sum of absolute values of its incident weights (excluding its local bias) is upper bounded by* $\Xi_p = \sum_{q=1}^{m} |v(q,p)| < \varepsilon 2^{1/\varepsilon}$.

PROOF. (Sketch.) The maximum value of Ξ_p among $p \in \mathcal{H}$ is reached by unit z_{n+1} of the highest order $n+1$ (except for $n = 1$ when $\Xi_{x_1} = 2^{n+1}(W + (12u+39)n + 8) + W - 7 + \varepsilon$ dominates), that is $\Xi_{z_{n+1}} = 2V_{n+1} - 16n + 2 + \varepsilon < 2V_{n+1}$. Parameter $V_{n+1} = 7^{n-1}(V_2 + 5) - 5$ with $V_2 = 2W + (12u + 39)n + 48$ is computed by induction on n in which recursive formulas $v(x'_k, x_1) = 2v(x'_{k-1}, x_1)$, $v(x_k, p) = 2v(x_{k-1}, p)$ for $p \in \mathcal{C}_{k-2} \setminus \{x_1\}$ $(k > 2)$, and Figures 1, 2 are employed. Hence, $\Xi_p < 12 \cdot 7^n \left(10n^2 w_{\max} + 2n w_{\max} + 3n + 2\right) < \varepsilon 2^{1/\varepsilon}$ by assumption on ε in Theorem 1. ☐

LEMMA 2.
1. *Let $p \in \mathcal{H}$ be a unit saturated at $b \in \{0,1\}$ with a* defect

$$\delta_p(t) = |y_p(t) - b| \,, \tag{13}$$

for the duration of a continuous time interval $\tau = [t_0, t_f]$ for some $t_0 \geq 0$. Then the state dynamics of p converging towards value b can be explicitly solved as

$$y_p(t) = \left| b - \delta_p e^{-(t-t_0)} \right| \tag{14}$$

for $t \in \tau$, where $\delta_p = \delta_p(t_0)$ is p's initial defect.
2a. *Let $Q \subseteq \mathcal{H}$ be a subset of units saturated for the duration of time interval $\tau = [t_0, t_f]$. Then the dynamics of the excitation $\xi_p(t)$ for any unit $p \in \mathcal{H}$ can be described as*

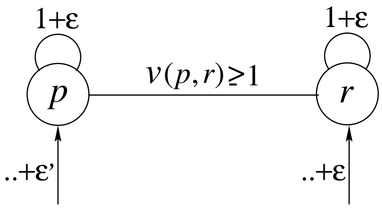
$$\begin{aligned} \xi_p(t) &= v(0,p) + \sum_{q \in Q;\, \xi_q(t) \geq 1} v(q,p) \\ &\quad + \sum_{q \notin Q} v(q,p) y_q(t) + \Delta_{pQ} e^{-(t-t_0)} \end{aligned} \tag{15}$$

for $t \in \tau$, where

$$\Delta_{pQ} = \sum_{q \in Q;\, \xi_q(t_0) \leq 0} v(q,p)\delta_p - \sum_{q \in Q;\, \xi_q(t_0) \geq 1} v(q,p)\delta_p \tag{16}$$

is the initial total weighted defect *of Q affecting $\xi_p(t_0)$.*
2b. *In addition, let $t_f > t_0 + t_1$ where*

$$t_1 = \frac{\ln 2}{\varepsilon} \,, \tag{17}$$



p	r	unit order
c_0	c_k	$1 \le k \le n+1$
c_k	a_k	$1 \le k \le n+1$
a_1	x_1	1
a_k	x'_k	$2 \le k \le n+1$
b'_k	x_k	$2 \le k \le n+1$
b_k	d_k	$1 \le k \le n+1$
d_k	z_k	$1 \le k \le n+1$
z_k	c_0	$1 \le k \le n+1$

Figure 4: Activity transfer from p to r in the clock network $\mathcal{C}$.

and assume that the respective weights in $\mathcal{H}$ satisfy either

$$v(0,p) + \sum_{q \in Q;\, \xi_q(t_0) \ge 1} v(q,p) + \sum_{q \notin Q;\, v(q,p) > 0} v(q,p) \;<\; -\varepsilon \qquad (18)$$

or

$$v(0,p) + \sum_{q \in Q;\, \xi_q(t_0) \ge 1} v(q,p) + \sum_{q \notin Q;\, v(q,p) < 0} v(q,p) \;>\; 1 + \varepsilon\,. \qquad (19)$$

Then p is saturated at either 0 or 1, respectively, for the duration of time interval $[t_0 + t_1, t_f]$.

LEMMA 3.
1. Consider a situation as depicted in Figure 4, where a clock unit $p \in \mathcal{C}$ with fractional part of bias $\varepsilon' \in \{\varepsilon, \varepsilon/3\}$ and feedback weight $v(p,p) = 1 + \varepsilon$ is supposed to activate and transfer a signal to the subsequent unit $r \in \mathcal{C}$ with bias fraction ε and $v(r,r) = 1 + \varepsilon$ via weight $v(p,r) \ge 1$. Let all the units incident on p, r excluding p, r be saturated for the duration of some sufficiently large time interval $\tau = [t_0, t_f]$ (e.g. $t_f > t_0 + t_2$ where t_2 is defined in (25)), starting at a time $t_0 \ge 0$ when $\xi_p(t_0) = 0$. Assume that the initial defects

$$\delta_p + \Delta_{rQ} < \varepsilon \qquad (20)$$

for $Q = \mathcal{H} \setminus \{p\}$ are bounded. Further assume that the respective weights satisfy

$$v(0,p) + \sum_{q \in Q;\, \xi_q(t_0) \ge 1} v(q,p) \;=\; \varepsilon' \qquad (21)$$

$$v(0,r) + \sum_{q \in Q;\, \xi_q(t_0) \ge 1} v(q,r) \;=\; \varepsilon - v(p,r)\,. \qquad (22)$$

Then p is unsaturated with the state dynamics

$$y_p(t) = \frac{\varepsilon' \left(e^{\varepsilon(t-t_0)} - 1\right)}{\varepsilon(1+\varepsilon)} - \frac{\varepsilon' + \Delta_{pQ} e^{-(t-t_0)}}{1+\varepsilon} \qquad (23)$$

exactly for the duration of time interval $(t_0, t_0 + t'_1)$, where

$$t'_1 = \frac{\ln\left(1 + \frac{\varepsilon}{\varepsilon'}\right)}{\varepsilon} \qquad (24)$$

(note $t'_1 = t_1$ for $\varepsilon' = \varepsilon$ and $t'_1 = 2t_1$ for $\varepsilon' = \varepsilon/3$), while r is saturated at 0. In addition, p is saturated at 1 for the duration of time interval $[t_0 + t'_1, t_f]$, while r unsaturates from 0 at time $t_0 + t_2$ where

$$t_2 = \ln \frac{v(p,r)\delta_p(t_0 + t'_1)\left(1 + \frac{\varepsilon}{\varepsilon'}\right)^{1/\varepsilon} - \Delta_{rQ}}{\varepsilon} \ge t'_1\,. \qquad (25)$$

2. Consider a situation as depicted in Figure 5, where a clock unit $p \in \mathcal{C}$ with bias $v(0,p) = -1 + \varepsilon$ and feedback weight $v(p,p) = 1 + \varepsilon$ is supposed to receive a signal from the preceding unit $o \in \mathcal{C}$, activate itself, and further transfer the signal to the subsequent unit $r \in \mathcal{C}$ with $v(0,r) = -1 + \varepsilon/3$ and $v(r,r) = 1 + \varepsilon$, via unit weight $v(p,r) = 1$, while a set $R \subseteq \mathcal{H}$ of units (define $B_{t_0} = \{q \in B;\, \xi_q(t_0) \ge 1\}$ in Figure 5), incident on p (with no connections to r) may unsaturate after p unsaturates from 0. Let all the units incident on p, r, excluding p, r, and R, be saturated for the duration of a sufficiently large time interval $\tau = [t_0, t_f]$ (e.g. at least until r unsaturates from 0) starting at a time $t_0 \ge 0$ when $\xi_p(t_0) = 0$. Assume that the initial defects

$$\delta_r \;<\; \varepsilon 2^{-1/\varepsilon} \qquad (26)$$

$$\Delta_{rQ'} \;<\; \varepsilon 2^{-1/\varepsilon} \qquad (27)$$

for $Q' = \mathcal{H} \setminus (R \cup \{p\})$ and also

$$(1+\varepsilon)\delta_p + \sum_{q \in R;\, \xi_q(t_0) \le 0} v(q,p)\delta_q - \sum_{q \in R;\, \xi_q(t_0) \ge 1} v(q,p)\delta_q \;\le\; \varepsilon 2^{-1/\varepsilon} \qquad (28)$$

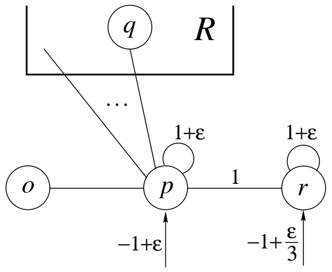
outside Q', are bounded. Further assume that the respective weights satisfy

$$v(0,p) + \sum_{q \in Q';\, \xi_q(t_0) \ge 1} v(q,p) + \sum_{q \in R;\, v(q,p) < 0} v(q,p) \;=\; \varepsilon \qquad (29)$$

$$\sum_{q \in Q';\, \xi_q(t_0) \ge 1} v(q,r) \;=\; 0\,. \qquad (30)$$

Then p saturates at 1 in time at most $t_0 + 2t_1$, remaining then saturated until time at least t_f, and r unsaturates from 0 only after p is saturated at 1.

PROOF. (Sketch.)
1. Excitation $\xi_p(t) = \varepsilon' + (1+\varepsilon)y_p(t) + \Delta_{pQ}e^{-(t-t_0)}$ of unit p for $t \in [t_0, t_0 + t_2]$ is obtained from formula (15) and assumption (21), which determines its state dynamics (5) by differential equation $(dy_p/dt)(t) = -y_p(t) + \varepsilon' + (1+\varepsilon)y_p(t) + \Delta_{pQ}e^{-(t-t_0)}$ when p is unsaturated. The corresponding initial condition $y_p(t_0) = (-\varepsilon' - \Delta_{pQ})/(1+\varepsilon) = \delta_p$ comes from $\xi_p(t_0) = 0$ which also bounds the initial defect as $-1 - \varepsilon - \varepsilon' \le \Delta_{pQ} \le -\varepsilon' < 0$, due to $1 \ge \delta_p \ge 0$. Hence solution (23) follows, which provides dynamics $\xi_p(t) = \varepsilon'(e^{\varepsilon(t-t_0)} - 1)/\varepsilon > 0$ for the excitation of unit p, ensuring that p is unsaturated exactly for the duration of $(t_0, t_0 + t'_1)$, even though



o	p	r	unit order	R
a_1	x_1	b_1	1	$\{c_0\} \cup A \cup B_{t_0}$
a_k	x'_k	b'_k	$2 \le k \le n+1$	$\{x_1\}$
b'_k	x_k	b_k	$2 \le k \le n+1$	$\mathcal{C}_{k-1} \setminus \{x_1\}$

Figure 5: Activity transfer from p to r when units $q \in R$ unsaturate.

its state $y_p(t)$ is initially decreasing for all $t \in (t_0, t_0 + t_g)$ where $t_g = (\ln(-\Delta_{pQ}/\varepsilon'))/(1+\varepsilon) < t'_1$. Similarly, excitation $\xi_r(t) = \varepsilon - v(p,r) + v(p,r)y_p(t) + \Delta_{rQ}e^{-(t-t_0)}$ of r is obtained from (15) and (22) which should prove to be non-positive for all $t \in (t_0, t_0 + t'_1)$. Since $v(p,r) \ge 1$, it suffices to show $\varepsilon + y_p(t) - 1 + \Delta_{rQ}e^{-(t-t_0)} \le 0$ that further reduces to

$$\begin{aligned}\varepsilon\left(\varepsilon' + \varepsilon(1+\varepsilon)\right)e^{-(t-t_0)} & \\ +\varepsilon'\left(e^{\varepsilon(t-t_0)} - 1\right) - \varepsilon \quad & \le \quad \varepsilon\left(\varepsilon' - \varepsilon^2\right) \end{aligned} \tag{31}$$

by using dynamics equation (23) in which $-\Delta_{pQ} = \varepsilon' + (1+\varepsilon)\delta_p$, and by assumption (20). For $t \in [t_0, t_0 + t_\varepsilon]$ where $t_\varepsilon = \ln((\varepsilon' + \varepsilon(1+\varepsilon))/(\varepsilon' - \varepsilon^2))$, term $e^{\varepsilon(t-t_0)}$ reaches its maximum at $t_0 + t_\varepsilon$ whereas $e^{-(t-t_0)} \le 1$ implying (31). For $t \in [t_0 + t_\varepsilon, t_0 + t'_1]$, term $\varepsilon(\varepsilon' + \varepsilon(1+\varepsilon))e^{-(t-t_0)}$ in (31) achieves its maximum $\varepsilon(\varepsilon' - \varepsilon^2)$ at $t_0 + t_\varepsilon$ while $\varepsilon'(e^{\varepsilon(t-t_0)} - 1) - \varepsilon \le 0$ reaching 0 at $t_0 + t'_1$. Hence, r is saturated for the duration of $(t_0, t_0 + t'_1)$. Furthermore, excitation $\xi_p(t) = 1 + (\varepsilon + \varepsilon')(1 - e^{-(t-t_0-t'_1)}) \ge 1$ of p saturated at 1, which can be derived from dynamics equation (14) with initial defect $\delta_p(t_0 + t'_1) = 1 - y_p(t_0 + t'_1) = (\varepsilon + \varepsilon' + \Delta_{pQ}(1 + \varepsilon/\varepsilon')^{-1/\varepsilon})/(1+\varepsilon)$ obtained from (23) and (24), ensures that p stays saturated at 1 at least for the duration of $[t_0 + t'_1, t_0 + t_2]$, where t_2 in (25) comes from $\xi_r(t_0 + t_2) = \varepsilon - v(p,r)\delta_p(t_0 + t'_1)e^{-(t_2 - t'_1)} + \Delta_{rQ}e^{-t_2} = 0$. Finally, it must also be checked that $\xi_p(t) = \varepsilon' + 1 + \varepsilon + v(p,r)y_r(t) - (1+\varepsilon)\delta_p(t_0+t_2)e^{-(t-t_0-t_2)} + (\Delta_{pQ} - v(p,r)\delta_r)\, e^{-(t-t_0)} \ge 1$ for all $t \in [t_0 + t_2, t_f]$. Here, $v(p,r)y_r(t) > 0$ whereas the respective defect terms having the least value at $t_0 + t_2$ can be lower bounded by $-\varepsilon' - \varepsilon$ when the explicit formulas are substituted for $\delta_p(t_0 + t_2)$, t_2, Δ_{pQ}, and inequalities (20), $\delta_r \le 1$, $v(p,r) \ge 1$, $\varepsilon' \ge \varepsilon/3$ are applied.

2. Notice that unit o saturates at 1 before p is unsaturated from 0 according to case 1 of this lemma. Excitation $\xi_p(t) \ge \varepsilon + (1+\varepsilon)y_p(t) + \Delta_{pQ'}\, e^{-(t-t_0)}$ of p from (15) for $t \in \tau$ is lower bounded by assumption (29). According to dynamics (5) this also provides a lower bound on its state derivative $(dy_p/dt)(t) \ge \varepsilon y_p(t) + \varepsilon + \Delta_{pQ'}\, e^{-(t-t_0)}$ when p is unsaturated. In the beginning of time interval τ, the state evolution of p is determined by equation (23) before the first $q \in R$ unsaturates, since assumption (21) coincides with (29) due to $\varepsilon' = \varepsilon$ and $\xi_q(t_0) \ge 1$ for all $q \in R$ with $v(q,p) < 0$ (see table in Figure 5). Hence, the initial defect $\Delta_{pQ'} = \Delta_{pQ} - \sum_{q\in R;\, \xi_q(t_0)\le 0} v(q,p)\delta_q + \sum_{q \in R;\, \xi_q(t_0)\ge 1} v(q,p)\delta_q$ is expressed in terms of $\Delta_{pQ} = -\varepsilon - (1+\varepsilon)\delta_p$ from case 1 of this Lemma by using definition (16) so that assumption (28) can be applied to lower bound

$$\Delta_{pQ'} \ge -\varepsilon\left(1 + 2^{-1/\varepsilon}\right) \tag{32}$$

which gives

$$\frac{dy_p}{dt}(t) \ge \varepsilon y_p(t) + \varepsilon - \varepsilon\left(1 + 2^{-1/\varepsilon}\right)e^{-(t-t_0)}. \tag{33}$$

Since $\varepsilon y_p(t) \ge 0$, it follows that $(dy_p/dt)(t) \ge \varepsilon - \varepsilon^2 > 0$ for $t \ge t_0 + t_d$ where $t_d = \ln((1 + 2^{-1/\varepsilon})/\varepsilon)$, provided that p is still unsaturated. This implies that $y_p(t)$ grows at least as fast as the straight line with equation $(\varepsilon - \varepsilon^2)(t - t_0 - t_d) - y = 0$ until p saturates at 1. Thus, p saturates at 1 certainly before $t_0 + t_d + t_s < t_0 + 2t_1$ where $t_s = 1/(\varepsilon - \varepsilon^2)$ because $\xi_p(t) > y_p(t)$ from dynamics (5) due to its state derivative being positive for $t \ge t_0 + t_d$. Similarly, excitation $\xi_r(t) = -1 + \varepsilon/3 + y_p(t) + \Delta_{rQ'}\, e^{-(t-t_0)}$ of r saturated at 0 is derived from (15) and (30). Let $t_y > 0$ be the least local time instant at which $y_p(t_0 + t_y) = 1 - \varepsilon/3 - \Delta_{rQ'}\, e^{-t_y}$ when r is still saturated at 0 since $\xi_r(t_0 + t_y) = 0$. By using (27) and (32), excitation $\xi_p(t_0 + t_y) \ge \varepsilon + (1+\varepsilon)(1 - \varepsilon/3 - \Delta_{rQ'}\, e^{-t_y}) + \Delta_{pQ'}\, e^{-t_y}$ of p at $t_0 + t_y$ can be lower bounded by 1, ensuring p is already saturated at 1 at $t_0 + t_y$. Finally, it must be checked that $\xi_p(t) \ge \varepsilon + (1+\varepsilon)(1 - (\varepsilon/3 + \Delta_{rQ'}e^{-t_y})e^{-(t-t_0-t_y)}) + y_r(t) + (\Delta_{pQ'} - \delta_r)e^{-(t-t_0)} \ge 1$ for all $t \in [t_0 + t_y, t_f]$ when r may unsaturate, which follows from $y_r(t) \ge 0$ and the respective defect bounds (26), (27), and (32). □

Lemma 4. *All the units in $A \cup B$ are saturated at 0 when the state of the clock interface unit $x_1 \in \mathcal{C}$ equals 2/3.*

***1.** The state of unit x_1 unsaturated from 0 at time $t_0 \ge 0$ after reaching value $y_{x_1}(t_0 + t'_\ell) = 2/3$ from below remains further increasing at least until the next x'_k unit ($2 \le k \le n+1$) is unsaturated from 0.*

***2.** Let unit x'_k ($2 \le k \le n+1$) cease its saturation at 0 at time $t_0 \ge 0$ when $\xi_{x'_k}(t_0) = 0$. Following this, let unit x_1 unsaturate from 1 immediately after time instant $t_0 + t_u > t_0$, where $\xi_{x_1}(t_0 + t_u) = 1$, and reach its state value $y_{x_1}(t_0 + t_\ell) = 2/3$ from above at time $t_0 + t_\ell$ ($t_u < t_\ell$). Let*

$$\Delta_{x_1 Q_1} > -\varepsilon \tag{34}$$

for $Q_1 = \mathcal{H} \setminus \{x'_k\}$ and

$$\Delta_{x_1 Q'_1} > -\varepsilon 2^{-1/\varepsilon} \tag{35}$$

for $Q'_1 = \mathcal{H} \setminus (A_{t_0} \cup B \cup \{x_1, x'_k\})$ where $A_{t_0} = \{q \in A;$

$\xi_q(t_0) \geq 1\}$. *Further assume that*

$$\delta_{x_1} < \varepsilon 2^{-1/\varepsilon} \quad (36)$$

$$\sum_{q \in A_{t_0}} v(q, x_1)\delta_q < \varepsilon 2^{-1/\varepsilon} \quad (37)$$

$$\sum_{q \in B} v(q, x_1)\delta_q > -\varepsilon 2^{-1/\varepsilon}. \quad (38)$$

Then the state of unit x_1 keeps decreasing after time $t_0 + t_\ell$ at least until x_1 is again unsaturated from 0 by a_1.

PROOF. (Sketch.) It can first be easily checked that $\xi_p \leq -2$ for all $p \in A \cup B$ when $y_{x_1} = 2/3$.

1. Obviously, $y_{x_1}(t)$ is further increasing for $t \geq t_0 + t_\ell$ since $(dy_{x_1}/dt)(t) > 0$ for $y_{x_1}(t_0 + t_\ell) = 2/3$ according to (33).

2. Excitation $\xi_{x_1}(t_0 + t_u) = W + (12u + 39)n + 3 + 2\varepsilon + v(x'_k, x_1)y_{x'_k}(t_0 + t_u) + \Delta_{x_1 Q_1} e^{-t_u} = 1$ of x_1 at $t_0 + t_u$ is derived from (15) where $v(0, x_1) + \sum_{q \in Q_1; \xi_q(t) \geq 1} v(q, x_1) = W + (12u + 39)n + 3 + 2\varepsilon$ follows from the construction of $\mathcal{H}$ (recall $\alpha_j \in A_{t_0} \longleftrightarrow \beta_j \notin A_{t_0}$). Hence, $y_{x'_k}(t_0 + t_u) \geq -(W + (12u + 39)n + 2 + \varepsilon)/v(x'_k, x_1)$ due to assumption (34). We already know from inequality (33) that $(dy_{x'_k}/dt)(t) \geq \varepsilon y_{x'_k}(t) - \varepsilon 2^{-1/\varepsilon}$. Since $y_{x'_k}$ is continuous, for proving $(dy_{x'_k}/dt)(t) > 0$ for $t \geq t_0 + t_u$ it suffice to show $(dy_{x'_k}/dt)(t_0 + t_u) \geq -\varepsilon(W + (12u + 39)n + 2 + \varepsilon)/v(x'_k, x_1) - \varepsilon 2^{-1/\varepsilon} > 0$ at $t_0 + t_u$ which is achieved by $-v(x'_k, x_1) < \varepsilon 2^{1/\varepsilon}$ from Lemma 1. Thus, the state of x'_k keeps increasing after x_1 unsaturates from 1. Similarly, excitation $\xi_{x_1}(t) = W + 1 + \varepsilon + (1 + \varepsilon)y_{x_1}(t) + v(x'_k, x_1)y_{x'_k}(t) + \sum_{q \in A_{t_0}} v(q, x_1)y_q(t) + \sum_{q \in B} v(q, x_1)y_q(t) + \Delta_{x_1 Q'_1} e^{-(t - t_0)}$ of x_1 for time $t \geq t_0$ is obtained from (15) where $v(0, x_1) + \sum_{q \in Q'_1; \xi_q(t) \geq 1} v(q, x_1) = W + 1 + \varepsilon$. Hence,

$$\begin{aligned} \frac{dy_{x_1}}{dt}(t) &= \varepsilon y_{x_1}(t) + v(x'_k, x_1)y_{x'_k}(t) \\ &\quad + \sum_{q \in A_{t_0}} v(q, x_1)y_q(t) + \sum_{q \in B} v(q, x_1)y_q(t) \\ &\quad + \Delta_{x_1 Q'_1} e^{-(t - t_0)} + W + 1 + \varepsilon \end{aligned} \quad (39)$$

of unsaturated x_1 for $t \geq t_0 + t_u$ according to (5). Denote by $0 < t'_d < t_\ell$ the least local time instant such that $(dy_{x_1}/dt)(t_0 + t'_d) = 0$ ($t'_d > t_u$). The change in values of particular terms in the state derivative (39) of x_1 will be estimated between time instants $t_0 + t'_d$ and $t_0 + t_\ell$. Thus, $y_{x_1}(t_0 + t'_d) > y_{x_1}(t_0) = 1 - \delta_{x_1} > 1 - \varepsilon 2^{-1/\varepsilon} > 2/3 = y_{x_1}(t_0 + t_\ell)$ by assumption (36) which implies

$$\varepsilon y_{x_1}(t_0 + t_\ell) - \varepsilon y_{x_1}(t_0 + t'_d) < -\frac{\varepsilon}{3} + \varepsilon^2 2^{-1/\varepsilon}. \quad (40)$$

Since $v(x'_k, x_1) < 0$ and $y_{x'_k}(t)$ is increasing for $t \geq t_0 + t_u$ it follows

$$v(x'_k, x_1)y_{x'_k}(t) - v(x'_k, x_1)y_{x'_k}(t_0 + t'_d) < 0 \quad (41)$$

for $t \geq t_0 + t'_d$. In addition, $v(q, x_1) > 0$ for $q \in A_{t_0}$ and all the units in A_{t_0} are still saturated at 1 at $t_0 + t'_d$ which gives

$$\sum_{q \in A_{t_0}} v(q, x_1)\left(y_q(t) - y_q(t_0 + t'_d)\right) < \varepsilon 2^{-1/\varepsilon} \quad (42)$$

for $t \geq t_0 + t'_d$ according to assumption (37). Similarly, $v(q, x_1) < 0$ for $q \in B$ and all the units in B are still saturated at 0 at $t_0 + t'_d$ which provides

$$\sum_{q \in B} v(q, x_1)\left(y_q(t) - y_q(t_0 + t'_d)\right) < \varepsilon 2^{-1/\varepsilon} \quad (43)$$

for $t \geq t_0 + t'_d$ by (38). Also the defect term in (39) satisfies

$$\Delta_{x_1 Q'_1} e^{-(t - t_0)} - \Delta_{x_1 Q'_1} e^{-t'_d} < \varepsilon 2^{-1/\varepsilon} \quad (44)$$

for $t \geq t_0 + t'_d$ according to (35). By summing inequalities (40)–(44) for particular term differences in (39) we obtain $(dy_{x_1}/dt)(t_0 + t_\ell) = (dy_{x_1}/dt)(t_0 + t_\ell) - (dy_{x_1}/dt)(t_0 + t'_d) < -\varepsilon/3 + \varepsilon(3 + \varepsilon)2^{-1/\varepsilon} < 0$ which is further valid even for $t \geq t_0 + t_\ell$ when x_1 is unsaturated since bounds (40)–(44) still apply and $y_{x_1}(t)$ is continuous. □

The correct timing of the simulation still needs to be verified to ensure a sufficiently fast decrease in the defects of the continuous-time correlates of binary states, because the analysis in Lemmas 3 and 4 is valid only if defect bounds (20), (26)–(28), and (34)–(38) are satisfied. According to Lemma 2.2b, the absolute value of the total weighted defect affecting any unit in Hopfield net $\mathcal{H}$ is bounded by ε after time t_1, decreasing further to $\varepsilon 2^{-1/\varepsilon}$ by time $2t_1$. On the other hand, t_1 represents a lower bound on the time necessary for activating a typical clock unit $p \in \mathcal{C}$ (see table in Figure 4) by Lemma 3.1. Hence, it can be shown from the clock dynamics that the subsequent clock unit $r \in \mathcal{C}$ in Lemma 3.1 has always time at least t_1 for decreasing the defect induced by its incident saturated units below ε as assumed in (20) even before unit p starts its activation. Similarly, stronger defect bounds (26)–(28) in Lemma 3.2 are met since time $2t_1$ is guaranteed for incident saturated units to decrease their defects before unit p unsaturates. It follows that both the phases of a simulated discrete step controlled by the clock interface unit x_1 take time at least, say $4t_1$. This suffices for the correct approximate simulation by gate subnetworks $\mathcal{G}_j$ according to Lemma 2.2b, and guarantees assumptions (34)–(38) of Lemma 4 for the correct synchronization. Also the lower bound $\Omega(t^*/\varepsilon)$ on the total simulation time follows immediately from the previous time analysis and equation (17). Further, every unsaturated unit in Hopfield net $\mathcal{H}$ saturates within time at most $3t_1$. Moreover, during the simulation all the units in $\mathcal{H}$ can simultaneously be saturated for a period of at most t_1 before the respective defects decrease below ε and the next unit unsaturates. This implies the corresponding upper bound $O(t^*/\varepsilon)$ and completes the proof of the theorem.

5. A SIMULATION EXAMPLE

A computer program HNGEN has been created to automate the construction from Theorem 1. The input for HNGEN is a text file containing the asymmetric weights and biases of a TLN, as well as its initial state. The program generates the corresponding continuous-time symmetric Hopfield system together with its initial conditions in the form of a FORTRAN subroutine. This FORTRAN procedure is then presented to a numerical solver from the NAG library that provides the user with a numerical solution for the respective system (5). By using the program HNGEN, the underlying construction has been successfully tested on several examples. Consider e.g. the simple 3-gate cycle TLN

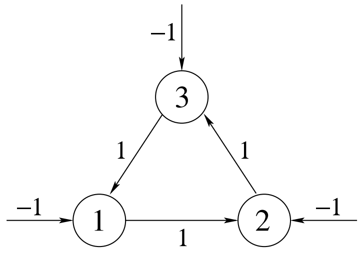


Figure 6: A three-gate cycle TLN.

depicted in Figure 6, initiated in a state where gate 1 has output 1 and gates 2 and 3 output 0. Then the computation of the network consists simply of propagating the unit signal around the cycle. Implementing this system on the HNGEN generator results in a continuous-time symmetric Hopfield system with 61 variables. Figure 7 shows the numerical evolution of the state variables corresponding to the clock interface unit x_1 and the three units π_1, π_2, π_3 that represent binary states of the original discrete gates, for a period of eight (2^3) simulated discrete steps. A parameter value of $\varepsilon = 0.3$ was used in this numerical simulation, showing that the theoretical estimate of ε in Theorem 1 is actually quite conservative.

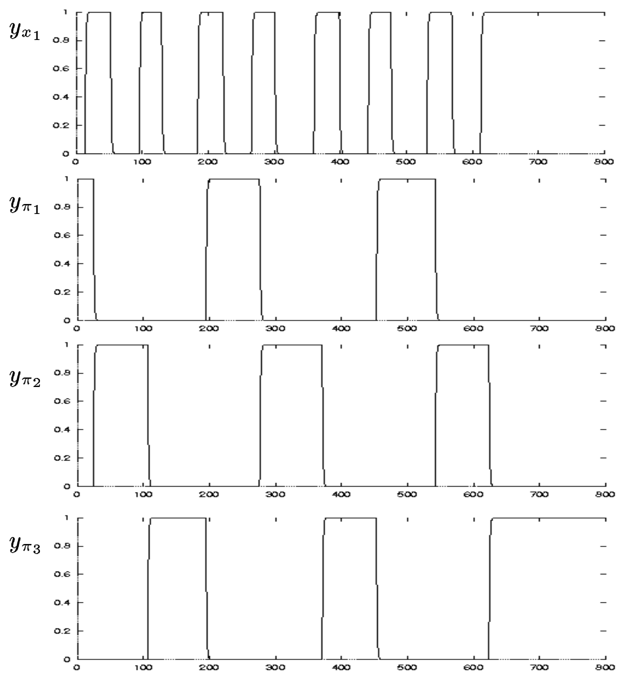


Figure 7: Continuous-time symmetric simulation of three-gate cycle TLN for $\varepsilon = 0.3$.

6. CONCLUSIONS AND OPEN PROBLEMS

We have proved that an arbitrary convergent discrete-time threshold logic network can be simulated by a symmetric continuous-time Hopfield system with only a linear increase in the system dimension. The existence of Liapunov functions for Hopfield systems precludes the use of unbounded oscillations in such a simulation; nevertheless we are able to base the construction on the bounded, but exponentially long sequence of pulses generated by the continuous-time clock subsystem.

From the point of view of understanding analog computation in general this technique is somewhat unsatisfying, since we are still basically discretizing the continuous-time computation. It would be most interesting to develop some theoretical tools (e.g. complexity measures, reductions, universal computation) for "naturally" continuous-time computations that exclude the use of discretizing oscillations.

Another challenge for further research is to prove *upper bounds* on the power of continuous-time systems. Note that in the case of discrete-time analog-state neural networks a single fixed-size network with rational-number parameters can be computationally universal, i.e. able to simulate a universal Turing machine on arbitrary inputs [27]. Can e.g. this strong universality result be generalized for continuous-time systems? Also, we have established an exponential lower bound on the convergence time of symmetric continuous-time systems [31]: can a matching upper bound be proved, or the lower bound be increased?

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